PLASMA AND DIELECTRIC BARRIER DISCHARGE ACTUATOR

RADAR CROSS SECTION

by

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Abstract

Plasma actuators for aerodynamic applications are receiving significant research attention and it is necessary to know the effects of this plasma on the vehicle radar cross section (RCS). This study identifies the critical parameters affecting the RCS of plasma along with experimental techniques and numerical techniques for determining the effects of plasma on RCS.

A review of plasma physics is presented along with the background of cavity design, cavity-waveguide coupling, choke design and the microwave perturbation technique. A cylindrical dielectric barrier discharge (DBD) plasma under five different pressures is generated in an evacuated glass tube. The microwave perturbation method is used to measure permittivity and loss factor of the plasma and then the plasma frequency, electron-neutral collision rate and electron density are determined for these five pressures. Simulations by a commercial microwave simulator are comparable to the experimental results which show little effect at sea level and increasing effects with increased elevation.

Plasma has a capability that its refractive index can be controlled by changing parameters such as the electron density profile, plasma frequency or collision rate so the RCS becomes controllable. However controlling the aforementioned parameters practically in order to decrease RCS is not as simple as the theoretical simulations. From the various methods of plasma generation, our focus is on the DBD because this method has shown benefits like aerodynamic drag reduction noticed in the last decade. The effect of a plasma slab on the RCS of a conductive sheet is investigated. The RCS is simulated and measured to verify the validity of the model used for simulations. Moreover, we
show the DBD actuator generates extra scattering because of Bragg diffraction a phenomenon not previously reported in the DBD literature.

A comprehensive study on dispersive media RCS using Finite-Difference Time-Domain (FDTD) method is done. The method used to generate the plane wave is the discrete plane wave (DPW) method. However, by plane wave we mean a wave packet which can behave like a plane wave not a real plane wave. A 12-layer split-field perfectly matched layer (PML) is used for the Absorbing Boundary Conditions (ABC). The dispersive media is modelled by shift-operator FDTD. Near-to-Far Field transformation (NTFT) is applied to calculate RCS in the far-field. This NTFT method is based on the surface equivalence theorem (Huygens’s principle). Finally the Fast Fourier Transform (FFT) is used to transform time domain signals to the frequency domain. Moreover, the simulation was extended for anisotropic dispersive media for two general profiles of exponential and polynomial. The results show that by choosing the right profile the RCS can be reduced to a large extent; however, achieving such a profile practically is very challenging.
Dedication

This dissertation is lovingly dedicated:

To my wife,

Naeeme Bidgol,

who has sacrificed so much for me.

To my mother and my father,

Farzaneh Daneshvar and Jalal Mirhosseini,

who have been role models to me all of my life.

To my sister and brother,

Elham and Farshid,

who have provided the extra motivation to finish my Ph.D.

To my daughter

Ariana Parvin

Who shined my life.

And finally to all my teachers,

who sparked a passion that has not died.

Their support, encouragement, and constant love have sustained me throughout my life.
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<td>2D</td>
<td>Two dimensional</td>
</tr>
<tr>
<td>3D</td>
<td>Three dimensional</td>
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<tr>
<td>ABC</td>
<td>Absorbing boundary condition</td>
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<tr>
<td>AFP</td>
<td>Analytic field propagator</td>
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<tr>
<td>B</td>
<td>Magnetic flux density (magnetic induction)</td>
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<tr>
<td>C</td>
<td>Capacitance</td>
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<tr>
<td>c</td>
<td>Speed of light</td>
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<tr>
<td>CCECE</td>
<td>Canadian Conference on Electric and Computer Engineering</td>
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<tr>
<td>cm</td>
<td>Centimeter</td>
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<tr>
<td>CST</td>
<td>Computer Simulation Technology</td>
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<tr>
<td>D</td>
<td>Electric displacement field (electric induction)</td>
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<td>DBD</td>
<td>Dielectric barrier discharge</td>
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<td>DPW</td>
<td>Discrete plane wave</td>
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<td>e</td>
<td>Proton charge</td>
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<td>E</td>
<td>Electric field</td>
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<td>eV</td>
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<td>f</td>
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<td>$f_s$</td>
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<td>FEM</td>
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<td>IFA</td>
<td>Incident field array</td>
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<tr>
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<td>Current density</td>
</tr>
<tr>
<td>$j$</td>
<td>Imaginary number used to represent $\sqrt{-1}$</td>
</tr>
<tr>
<td>$\vec{k}$</td>
<td>Wave vector</td>
</tr>
<tr>
<td>$k_{fd}$</td>
<td>Imaginary part of refractive index</td>
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<td>L</td>
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<td>Local Thermodynamic Equilibrium</td>
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<td>O-AFP</td>
<td>Optimized analytic field propagator</td>
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<td>( P_0 )</td>
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<td>( P'_{\zeta,\psi} )</td>
<td>E-field polarization projection</td>
</tr>
<tr>
<td>PEC</td>
<td>Perfect electric conductor</td>
</tr>
<tr>
<td>PML</td>
<td>Perfectly matched layer</td>
</tr>
<tr>
<td>PO</td>
<td>Physical optics</td>
</tr>
<tr>
<td>( q )</td>
<td>Charge of particle</td>
</tr>
<tr>
<td>( Q_0 )</td>
<td>Cavity unperturbed quality factor</td>
</tr>
<tr>
<td>( Q_1 )</td>
<td>Cavity perturbed quality factor</td>
</tr>
<tr>
<td>( q_T )</td>
<td>Test particle charge</td>
</tr>
<tr>
<td>( q_s )</td>
<td>Charge of species ( s )</td>
</tr>
<tr>
<td>( r )</td>
<td>Distance</td>
</tr>
<tr>
<td>( \hat{r} )</td>
<td>Unit vector</td>
</tr>
<tr>
<td>( \dot{r} )</td>
<td>First derivative of ( r )</td>
</tr>
<tr>
<td>( r_0 )</td>
<td>Radius of a circular aperture</td>
</tr>
<tr>
<td>( r_g )</td>
<td>Vector pointing to the particle from the center of the orbit</td>
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<tr>
<td>RCS</td>
<td>Radar cross section</td>
</tr>
<tr>
<td>S</td>
<td>Species</td>
</tr>
<tr>
<td>SF</td>
<td>Split field</td>
</tr>
<tr>
<td>SO</td>
<td>Shift operator</td>
</tr>
</tbody>
</table>
t  Time

$T_i$  Ion Temperature

TF/SF  Total-Field/Scattered-Field

$v_\perp$  Perpendicular speed

$v_0$  Incident speed

$v_s$  Thermal speed

VB  Visual basic

VNA  Vector network analyzer

$\bar{X}_L$  Reactance

$Y_{in}$  Input normalized admittance

$Y_\omega$  Admittance at frequency $\omega$

$Z_e$  Ion charge

$Z_\omega$  Impedance at frequency $\omega$

$\alpha_e$  Electric polarizability

$\alpha_m$  Magnetic polarizability

$\beta$  Wave number

$\delta$  Distance between two particle

$\delta f$  Small change in parameter $f$

$\varepsilon$  Permittivity

$\varepsilon'$  Real part of permittivity

$\varepsilon''$  Imaginary part of permittivity

$\varepsilon_0$  Permittivity of free space

$\lambda_0$  Waveguide wavelength
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>$\lambda_D$</td>
<td>Debye length</td>
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<tr>
<td>$\lambda_e$</td>
<td>Electron Debye length</td>
</tr>
<tr>
<td>$\lambda_c$</td>
<td>Cutoff wavelength</td>
</tr>
<tr>
<td>$\lambda_s$</td>
<td>Debye length for the species $s$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>Number of Coulomb collisions</td>
</tr>
<tr>
<td>$\Lambda_g$</td>
<td>Distance between to array elements</td>
</tr>
<tr>
<td>$\Lambda_s$</td>
<td>Plasma parameter of species $s$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Mobility tensor</td>
</tr>
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<td>$\mu_0$</td>
<td>Free space permeability</td>
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<tr>
<td>$\nu$</td>
<td>Collision frequency</td>
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<td>$\nu_c$</td>
<td>Small-angle collision rate</td>
</tr>
<tr>
<td>$\nu_L$</td>
<td>Large-angle collision rate</td>
</tr>
<tr>
<td>$\nu_m$</td>
<td>Electron-Ion collision frequency for particle with mass $m$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Free charge density</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Conductivity tensor</td>
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<td>$\Omega$</td>
<td>Cavity complex resonance frequency</td>
</tr>
<tr>
<td>$\sigma_m$</td>
<td>Magnetic conductivity</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Average potential energy</td>
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<tr>
<td>$\chi_1$</td>
<td>Real part of susceptibility</td>
</tr>
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<td>$\chi_2$</td>
<td>Imaginary part of susceptibility</td>
</tr>
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<td>$\varphi$</td>
<td>Electric potential</td>
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<tr>
<td>$\omega$</td>
<td>Angular frequency</td>
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<tr>
<td>$\omega_{ce}$</td>
<td>Electron cyclotron frequency</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------</td>
</tr>
<tr>
<td>(\omega_{ci})</td>
<td>Ion cyclotron frequency</td>
</tr>
<tr>
<td>(\omega_{co})</td>
<td>Waveguide cut-off frequency</td>
</tr>
<tr>
<td>(\omega_e)</td>
<td>Electron plasma frequency</td>
</tr>
<tr>
<td>(\omega_i)</td>
<td>Ion plasma frequency</td>
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<tr>
<td>(\omega_p)</td>
<td>Total plasma frequency</td>
</tr>
<tr>
<td>(\omega_s)</td>
<td>Plasma frequency for species s</td>
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<tr>
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<tr>
<td>(\nabla)</td>
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</tr>
<tr>
<td>(\nabla \cdot)</td>
<td>Divergence</td>
</tr>
<tr>
<td>(\nabla \times)</td>
<td>Curl</td>
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</tbody>
</table>
Chapter 1- Introduction

1.1 Overview

The radar scattering properties of plasma form the basis of this dissertation which presents contributions in three distinct areas: understanding plasma physics in the context of interaction with microwaves, radar cross section (RCS) prediction through numerical simulation and the development of new RCS measurement techniques. RCS defines the effective area of an object intercepting an amount of incident power that if scattered isotropically produces the same level of power at the radar as the actual target. The RCS of a target is not the same as the physical cross-sectional area of the target.

In fact, the work began as an investigation of the radar scattering properties of dielectric barrier discharge (DBD) actuators. DBD actuators have found many applications in aeronautics such as drag reduction and turbulence control [1-3]. For the first phase of the research, the RCS of a DBD actuator at atmospheric pressure was measured and the results revealed approximately no change in RCS either with or without the plasma present. To verify this observation, we needed to know the plasma parameters (electron density, plasma electron frequency, and collision rate) at different pressures corresponding to different altitudes in order to simulate the RCS at different altitudes. The literature revealed no consensus on the plasma parameters particularly for lower pressures or higher altitudes. Most research has been conducted at atmospheric pressures and for non-air gases (Helium or Argon) using methods including Langmuir probe, interferometry or spectroscopy measurement [4-6]. Besides, these methods reported a wide range of electron density ($10^{11}$ to $10^{19}$) even for the same atmospheric pressure [7],[8]. In order to deal with this issue, a highly sensitive novel method using the cavity
perturbation technique was developed. By measuring plasma parameters, the RCS of the plasma slab, the copper plate covered by plasma and DBD actuators were simulated.

When working on RCS measurements, another phenomenon affecting RCS was realized. This phenomenon occurs due to the periodic structure of the DBD actuators and is known as Bragg diffraction. Bragg’s law explains why the periodic structures like crystals appear to reflect electromagnetic waves at certain angles of incidence. This phenomenon causes an increase in RCS for some specific intersection angles.

The traditional measure of an object's scattering behavior is the RCS pattern which plots the scattered field magnitude as a function of aspect angle for a particular frequency and polarization. Although suitable to calculate the power received by a radar operating with those particular parameters, the RCS pattern is an incomplete descriptor of the object's scattering behavior. While the RCS pattern indicates the effect of the scattering mechanism, it does not reveal the physical processes which cause the observed effect. In contrast, imaging techniques, which exploit frequency and angle diversity to spatially resolve the reflectivity distribution of complex objects, allow the association of physical features with scattering mechanisms. These processes, therefore, indicate the causal components of the overall signature level observed in RCS patterns. There are a few methods of RCS prediction. The methods of interest are those that can be applied to arbitrary three-dimensional targets. The methods most commonly encountered are physical optics (PO), microwave optics (ray tracing), method of moments (MoM), finite element method (FEM), and Finite-Difference Time-Domain method (FDTD). Each of these methods has its own pros and cons. For this research, a MATLAB® program was written using the FDTD method to combine the discrete plane wave (DPW) technique
and the shift operator (SO). This method shows high stability and error controllability for dispersive media simulations.

1.2 Contributions

This thesis developed the cavity perturbation technique to measure more precisely the values of different plasma parameters (plasma frequency, collision rate and electron density) at different pressures [9]. For this purpose, the cavity perturbation method was employed using a novel method of measurement by synchronization between a vector network analyser (VNA) and the plasma ignition pulse. Different methods of the microwave energy coupling through apertures were investigated and a specific shape of orifice was designed to couple energy between the waveguide and the cavity with maximum coupling factor. Chokes are designed and fabricated in order to prevent microwave power leakage through the holes in the cavity walls which are made to accommodate high-voltage probes for plasma generation.

The impact of plasma on the RCS of an object was investigated using these measured parameters. In other words, we investigate to what extent the RCS of a target can be reduced by changing the plasma parameters to make the target invisible to radar with which the word stealth has been associated since the early 1970s. The effect of a plasma layer on the RCS of a copper plate is measured and simulated [10]. It reveals that high pressure plasma does not affect RCS because of the low electron density if an alternating current (AC) voltage with constant magnitude is applied. By increasing the plasma maintenance voltage one can increase electron density, but that requires more power and stronger wire insulation. The experimental measurements and simulations also show that thin layers of plasma have a very small effect on the RCS while thicker layers will
decrease the RCS to a large extent. However, it is not presently feasible to cover a large target with a thick plasma layer.

Bragg diffraction was identified as a source of increased RCS in DBD actuators for some specific angles during the experimental tests and this concept has not been previously explored in the literature. The RCS of DBD actuators at atmospheric pressure was measured and the experiments revealed no significant change in the RCS of a conductive layer occurs when it is covered by a DBD actuator except for the effect of Bragg diffraction.

A computer program was developed using the FDTD method to investigate the combination of shift-operator and discrete plane wave methods for RCS simulation from dispersive media [11], [12]. This method shows high stability in simulation of electromagnetic waves in dispersive media and error controllability. The isolation between scattered-field region and total-field region is 300 dB.

1.3 Thesis Organization

The organization of this thesis is outlined below.

Chapter 1 provides an introduction as well as an outline for the thesis work.

Chapter 2 introduces the relevant plasma physics and microwave background concerning this thesis. In particular, it discusses the plasma parameters, Klimontovich and plasma fluid equation, cavity perturbation theory, waveguide-cavity coupling methods and choke design.

Chapter 3 discusses the application of the cavity perturbation method associated with a novel experimental technique to measure electron density, plasma frequency and
collision rate of the plasma at different pressures. Since the objects defined for the VNA are in visual basic (VB), for the experimental technique code was developed in VB to automate capturing data from a vector network analyser. The simulations which are in good agreement with experimental results are performed using CST-MWS (a commercial software used for microwave simulations). This chapter was based on a paper that has been published in Applied Physics Letters [9].

Chapter 4 uses the plasma parameters measured in Chapter 3 to simulate the RCS of a flat copper plate covered by a plasma layer. Another phenomenon known as Bragg diffraction is investigated to show that scattering from DBD actuators increases in some specific angles of incidence because of the periodic structure of the actuators. Measured results confirm these simulations. This chapter was based on a paper that has been published in the IEEE Transactions on Plasma Science [10].

Chapter 5 develops the FDTD method and perfectly matched layers to simulate RCS of dispersive media. The discrete plane wave and shift operator methods are combined and reveal good stability. This chapter was based on a paper that was presented and published in the 28th annual IEEE Canadian Conference on Electrical and Computer Engineering (CCECE) [11], and in Progress in Electromagnetics Research M [12].

Chapter 6 provides the discussion and conclusions of this thesis. The contributions of this thesis are summarized and recommendations for future work are provided.

This thesis is written in the form of a thesis with chapters as papers. The content for chapters two, three and four has been organized such that consecutive topics build on each other and flow naturally between successive chapters. Chapters two and five can be considered independently; however these two chapters are in conjunction with other
chapters and the whole chapters embodying a union. The author of this thesis has performed all simulations, fabrications, experiments, data analysis and wrote the corresponding articles of which he is the first author.
1.4 References


Chapter 2-Plasma Physics

2.1 Introduction

The plasma has been called the fourth state of matter because heating a solid makes a liquid, heating a liquid makes a gas, and heating a gas makes plasma. (Compare the ancient Greeks’ earth, water, air and fire.) The word plasma comes from the Greek plasma, meaning “something formed or molded”. However plasma is not just a result of phase transition and it can be generated using non-thermal ionization. It was introduced to describe ionized gases by Tonks and Langmuir [1], [2]. More than 99% of the known universe is in the plasma state. (Note that our definition excludes certain configurations such as the electron gas in metals and so-called “strongly coupled” plasmas which are found, for example near the surface of the sun).

In our definition of plasma, we shall always consider plasma having roughly equal numbers of singly charged ions (+e) and electrons (-e), each with average density $n_0$ (particles per cubic centimeter) plus charge neutral particles. In nature many plasmas have more than two species of charged particles, and many ions have more than one electron missing. Sections 2.3, 2.4 and 2.5 discuss the plasma kinetic and plasma fluid equation which results in the proof of the plasma permittivity tensor; so if the reader is not interested in the proof, these sections can be skipped and (2-133) can be used as the plasma permittivity for the remainder of the thesis in the experimental tests and the simulations.
2.2 Plasma Parameters

2.2.1 Debye Shielding

In plasma we have many charged particles flying around at high speeds. Consider a test particle of charge \( q_T > 0 \) and infinite mass is located at the origin of a three-dimensional coordinate system containing an infinite, uniform plasma. The test charge repels all other ions, and attracts all electrons. Thus, around our test charge the electron density \( n_e \) increases and the ion density decreases. The test ion gathers a shielding cloud that tends to cancel its own charge.

Consider Poisson’s equation relating the electric potential \( \varphi \) to the charge density \( \rho \) due to electrons, ions, and test charge [3],

\[
\nabla^2 \varphi = -4\pi \rho = 4\pi e(n_e - n_i) - 4\pi q_T \delta(r).
\]  

(2-1)

Where \( \delta(r) \equiv \delta(x)\delta(y)\delta(z) \) is the product of three Dirac delta function. After introducing the test charge, we wait for a long enough time that the electrons with temperature \( T_e \) have come to thermal equilibrium with themselves, and the ions with temperature \( T_i \) have come to thermal equilibrium with themselves, but not so long that the electrons and ions have come to thermal equilibrium with each other at the same temperature. Then equilibrium statistical mechanics predicts that

\[
\begin{align*}
  n_e &= n_0 \exp \left( \frac{e\varphi}{T_e} \right), \quad \text{and} \quad n_i &= n_0 \exp \left( \frac{-e\varphi}{T_i} \right).
\end{align*}
\]  

(2-2)

Each density becomes \( n_0 \) at large distances from the test charge where the potential vanishes. The equations of (2-2) just hold for a short period of time because the thermal equilibrium is an unstable condition. Boltzmann’s constant is absorbed into the temperatures \( T_e \) and \( T_i \), which have units of energy and are measured in units of electron-
volts (eV). Assuming that $e\varphi/T_e \ll 1$ and $e\varphi/T_i \ll 1$, we expand the exponents in (2-2) and write (2-1) away from $r = 0$ as

$$\nabla^2 \varphi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\varphi}{dr} \right) = 4\pi n_0 e^2 \left( \frac{1}{T_e} + \frac{1}{T_i} \right) \varphi. \quad (2-3)$$

If we define the electron and ion Debye lengths as

$$\lambda_{e,i} = \left( \frac{T_{e,i}}{4\pi n_0 e^2} \right)^{1/2}, \quad (2-4)$$

then the total Debye length would be

$$\lambda_D^{-2} = \lambda_e^{-2} + \lambda_i^{-2}. \quad (2-5)$$

Equation (2-3) then becomes

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\varphi}{dr} \right) = \lambda_D^{-2} \varphi. \quad (2-6)$$

Trying a solution of the form $\varphi = \bar{\varphi}/r$, we find

$$\frac{d^2 \bar{\varphi}}{dr^2} = \lambda_D^{-2} \bar{\varphi}. \quad (2-7)$$

The solution that falls off properly at large distances is $\bar{\varphi} \propto exp(-r/\lambda_D)$. From elementary electricity and magnetism we know that the solution to (2-1) at locations very close to $r=0$ is $\varphi = q_T/r$; thus, the desired solution to (2-1) at all distances is

$$\varphi = \frac{q_T}{r} exp \left( \frac{-r}{\lambda_D} \right). \quad (2-8)$$

The potential due to a test charge in plasma falls off much faster than in vacuum. This phenomenon is known as Debye shielding, and is our first example of plasma collective behavior. For distances $r \gg \lambda_D$ (the Debye length), the shielding cloud effectively cancels the test charge ($q_T$). Numerically, the Debye length of species $s$ with temperature $T_s$ is roughly $\lambda_s \approx 740[T_s(eV)/n(cm^{-3})]^{1/2}$ in units of cm.
2.2.2 Plasma Parameter

In a plasma where each species has a density of \( n_0 \), the distance between a particle and its nearest neighbor is roughly \( n_0^{-1/3} \). The average potential energy \( \Phi \) of a particle due to its nearest neighbor is, in absolute value,

\[
|\Phi| \sim \frac{e^2}{r} \sim n_0^{1/3} e^2.
\]  

(2-9)

Our definition of plasma requires that this potential energy be much less than the typical particle’s kinetic energy

\[
\frac{1}{2} m_s \left\langle v^2 \right\rangle = \frac{3}{2} T_s = \frac{3}{2} m_s v_s^2,
\]  

(2-10)

where \( m_s \) is the mass of species \( s \), \( \left\langle \quad \right\rangle \) means an average over all particle velocity at a given point in space, and we have defined the thermal speed \( v_s \) of species \( s \) by

\[
v_s \equiv \left( \frac{T_s}{m_s} \right)^{1/2}.
\]  

(2-11)

For electrons, \( v_e \approx 4 \times 10^7 T_e^{1/2} \) cm/s, when \( T_e \) is in units of electron-volt (eV). Our definition of plasma requires

\[
n_0^{1/3} e^2 \ll T_s,
\]  

(2-12)

or

\[
n_0^{2/3} \left( \frac{T_s}{n_0 e^2} \right) \gg 1.
\]  

(2-13)

Raising each side of (2-13) to the 3/2 power, and recalling the definition (2-4) of the Debye length, we have (dropping factors of \( 4\pi \), etc.)

\[
\Lambda_s = n_0 \lambda_s^3 \gg 1.
\]  

(2-14)

Where \( \Lambda_s \) is called the plasma parameter of species \( s \). (Note: some authors call \( \Lambda_s^{-1} \) the plasma parameter.) The plasma parameter is just the number of particles of species \( s \) in a box each side of which has length equal to the Debye length (a Debye cube). Equation (2-
14) tells us that, by definition, a plasma is an ionized gas that has many particles in a Debye cube. Numerically, $A_s \approx 4 \times 10^8 T_s^{3/2} (eV)/n_0^{1/2} (cm^{-3})$. We will often substitute the total Debye length $\lambda_D$ in (2-14), and define the result $\Lambda \equiv n_0 \lambda_D^3$ to be the plasma parameter.

### 2.2.3 Plasma Frequency

Consider a hypothetical slab of plasma of thickness $L$, $(0 \leq x \leq L)$, where for the present we consider the ions to have infinite mass, but equal density $n_0$ and opposite charge to the electrons while the electrons are held rigidly in place with respect to each other, but can move freely through the ions. Suppose the electron slab is displaced a distance $\delta$ to the right of the ion slab, and then allowed to move freely. An electric field will be set up, causing the electron slab to be pulled back toward the ions. When the electrons exactly overlap the ions, the net force is zero, but the electron slab has substantial speed to the left. Thus, the electron slab overshoots, and the net result is harmonic oscillation. The frequency of the oscillation is called the electron plasma frequency. It depends only on the electron density, the electron charge, and the electron mass. The electron plasma frequency is calculated as follows [3].

Gauss’s law in one dimension is $(\partial_x E \equiv \partial / \partial x)$

$$\partial_x E = \frac{\rho}{\varepsilon}.$$  \hspace{1cm} (2-15)

Where, $E$ is the electric field. We take the boundary condition $E(x = 0) = 0$, and assume throughout that $\delta \ll L$. From (2-15) the electric field over most of the slab is $4\pi n_0 e \delta$, and the force per unit area on the electron slab is
(electric field) \times (charge per unit area) or \(-4\pi n_0^2 e^2 \delta L\). Newton’s second law is
(force per unit area) = (mass per unit area) \times (acceleration), or
\((-4\pi n_0^2 e^2 \delta L) = (n_0 m_e L)(\ddot{\delta}),\) \hspace{1cm} (2-16)
where an overdot is a time derivative. Equation (2-16) is in the standard form of a harmonic oscillator equation,
\[ \ddot{\delta} + \left(\frac{4\pi n_0 e^2}{m_e}\right) \delta = 0, \] \hspace{1cm} (2-17)
with characteristic frequency
\[ \omega_e = \left(\frac{4\pi n_0 e^2}{m_e}\right)^{1/2}, \] \hspace{1cm} (2-18)
Where \(\omega_e\) is called the electron plasma frequency. Numerically, \(\omega_e = 2\pi \times 9000n_e^{1/2}\) s\(^{-1}\), where \(n_e\) is in the units of cm\(^{-3}\). By analogy with the electron plasma frequency (2-18) we define the ion plasma frequency \(\omega_i\) for a general ion species with density \(n_i\) and ion charge \(Z_e\) (\(Z\) is the number of electrons) as
\[ \omega_i = \left(\frac{4\pi n_i Z_e^2 e^2}{m_i}\right)^{1/2}. \] \hspace{1cm} (2-19)
The total plasma frequency \(\omega_p\) for two-component plasma is defined as
\[ \omega_p^2 \equiv \omega_e^2 + \omega_i^2. \] \hspace{1cm} (2-20)
Since electron mass is much smaller than ion mass \((m_e \ll m_i)\), for most plasmas in nature, \(\omega_e \gg \omega_i\), so \(\omega_p^2 = \omega_e^2\). The general response of un-magnetized plasma to a perturbation in the electron density is a set of oscillations with frequencies very close to the electron plasma frequency \(\omega_e\). The relation among the Debye length \(\lambda_s\), the plasma frequency \(\omega_s\), and the thermal speed \(v_s\), for the species \(s\), is
\[ \lambda_s = v_s / \omega_s. \] \hspace{1cm} (2-21)
### 2.2.4 Collisions

A typical charged particle in plasma is at any instant interacting electrostatically with many other particles. If we did not know about Debye shielding, we might think that a typical particle is simultaneously having Coulomb collisions with all of the other particles in the plasma. Coulomb collisions are elastic collisions between particles because of their own electric charge or electric field. However, the field of our typical particle is greatly reduced from its vacuum field at distances greater than a Debye length, so that the particle is really not colliding with particles at large distances. Thus, we may roughly think of each particle as undergoing $A$ simultaneous Coulomb collisions [3].

From our definition of plasma, we know that the potential energy of interaction of each particle with its nearest neighbor is small. Since the potential energy is a measure of the effect of a collision, this means that the strongest one of its $A$ simultaneous collisions (the one with its nearest neighbor) is relatively weak. Thus, a typical charged particle in a plasma is simultaneously undergoing $A$ weak collisions. We shall soon see that even though $A$ is a large number for plasma, the total effect of all the simultaneous collisions is still weak. Of course, a weak effect can still be a very important effect. In the magnetic bottles like Tokamaks currently used to study controlled thermonuclear fusion plasmas, ion-ion collisions are one of the most important loss mechanisms [4].

Mathematically, the importance of collisions is contained in an expression called the collision frequency, which is the inverse of the time it takes for a particle to suffer a collision. Exactly what is meant by a collision of a charged particle depends upon the definition, and we will consider two different definitions with different physical content. The mathematical derivation of the collision frequency is an approximate one, intended
to be simple but yet to yield the correct results within factors of two or so. A more rigorous development can be found in [5], [6].

Consider the situation shown in Figure 2.1. A particle of charge \( q \), mass \( m \), is incident on another particle of charge \( q_0 \) and infinite mass with incident speed \( v_0 \). If the incident particle were un-deflected, it would have position \( x = v_0 t \) along the upper dashed line in Figure 2.1, being at \( x = 0 \) directly above the scattering charge \( q_0 \) at \( t = 0 \). The separation \( p \) of the two dashed lines is the impact parameter. If the scattering angle is small, the final parallel speed (parallel to the dashed line) will be quite close to \( v_0 \). The perpendicular speed \( v_\perp \) can be obtained by calculating the total perpendicular impulse.

\[
m v_\perp = \int_{-\infty}^{\infty} dt F_\perp(t).
\]  

(2-22)

Where \( F_\perp \) is the perpendicular force that the particle experiences in its orbit. Since the scattering angle \( v_\perp / v_0 \) is small, we can to a good approximation use the unperturbed orbit \( x = v_0 t \) to evaluate the right side of equation (2-22). This approximation is a very useful one in plasma physics. In Figure 2.1, Newton’s second law with the Coulomb force law is

\[
m \ddot{r} = \frac{q q_0}{r^2} \hat{r},
\]  

(2-23)
where \( \hat{r} \) is a unit vector in \( r \)-direction. Then
\[
F_\perp = \frac{q_q}{r^2} \sin \theta = \frac{q_q \sin \theta}{(p/\sin \theta)^2} = \frac{q_q}{p^2} \sin^3 \theta.
\]
(2-24)

Where \( p = r \sin \theta \) since the particle is assumed to be travelling along the upper dashed line. Equation (2-22) then reads
\[
v_\perp = \frac{q_q}{m p^2} \int_{-\infty}^{\infty} d t \sin^3 \theta (t).
\]
(2-25)

The relation between \( \theta \) and \( t \) is obtained from
\[
x = -r \cos \theta = \frac{-p \cos \theta}{\sin \theta} = v_0 t,
\]
(2-26)

so that
\[
dt = \frac{p}{v_0 \sin^2 \theta} \cdot 
\]
(2-27)

Using (2-27) in (2-25), one can find
\[
v_\perp = \frac{q_q}{m p v_0} \int_0^{\pi} \sin \theta \left( \frac{2q_q}{mv_0 p} \right).
\]
(2-28)

A new parameter \( p_0 \) is defined. This parameter is called the Landau length
\[
p_0 = \frac{2q_q}{mv_0^2}.
\]
(2-29)

It results in
\[
\frac{v_\perp}{v_0} = \frac{p_0}{p},
\]
(2-30)

which is strictly valid only when \( v_\perp \ll v_0, p \gg p_0 \).

If \( q q_0 > 0 \) then \( p_0 \) is the distance of closest possible approach for a particle of initial speed \( v_0 \). Although (2.30) is not valid for large angle collision, let us use it to get a rough idea of the impact parameter \( p \) which yields a large collision; we do this by setting \( v_\perp \) equal to \( v_0 \) in (2-30) to obtain \( p = p_0 \). Thus, any impact parameter \( p \leq p_0 \) will yield a large angle collision. Suppose the incident particle is an electron, and the (almost)
stationary scatterer is an ion. (Although Figure 2.2 shows a repulsive collision, this development is equally valid for attractive collisions.) The cross section for scattering through a large angle by one ion is \( \pi p_0^2 \). Consider an electron that enters a gas of ions. It will have a large angle collision after a time given roughly by setting (the total cross section of the ions in a tube of unit cross-sectional area, and length equal to the distance traveled) equal to (the unit area), or (time) \( \times \) (velocity) \( \times \) (number per unit volume) \( \times \) (cross section) = 1. The inverse of this time gives us the collision frequency \( \nu_L \) for large angle collisions; thus

\[
\nu_L = \pi n_0 v_0 p_0^2 = \frac{4\pi n_0 q^2 q_0^2}{m^2 v_0^2} = \frac{4\pi n_0 e^4}{m^2 v_0^2}. \tag{2-31}
\]

Note, \( \nu_L \) is proportional to the inverse of the third power of the particle speed. Recall that a typical charged particle in plasma is simultaneously undergoing \( \Lambda \) collisions. Only a very few of these are of the large angle type that lead to (2-31). Since a large angle collision involves a potential energy of interaction comparable to the kinetic energy of the incident particle and, by the definition of plasma, the potential energy of a particle due to its nearest neighbor is small compared to its kinetic energy. Thus, a particle undergoes many more small-angle collisions than large angle collisions. It turns out the cumulative effect of these small angle collisions is substantially larger than the effect of the large angle collisions, as we shall now show.

Unlike the large angle collisions, the many small angle collisions can produce a large effect only after many of them occur, but these small angle collisions produce velocity changes in random directions, some up, some down, some left, some right. We need to know how to measure the cumulative effect of many small random events.
Consider a variable $\Delta x$ that is the sum of many small independent random variables $\Delta x_i$, $i=1,2,\ldots,N$,

$$\Delta x = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_N, \quad (2-32)$$

Suppose $\langle \Delta x_i \rangle = 0$ for each $i$ and $\langle (\Delta x_i)^n \rangle$ is the same for each $i$, where $\langle \quad \rangle$ indicates ensemble average. Furthermore, suppose $\langle \Delta x_i \Delta x_j \rangle = 0$ if $i \neq j$, so that $\Delta x_i$ is uncorrelated with $\Delta x_j$, $i \neq j$. Then by (2-32) we have $\langle \Delta x_i \rangle = 0$, and

$$\langle (\Delta x)^2 \rangle = \langle (\sum_{i=1}^{N} \Delta x_i)^2 \rangle = \sum_{i=1}^{N} \langle (\Delta x_i)^2 \rangle = N \langle (\Delta x_i)^2 \rangle. \quad (2-33)$$

Consider a typical particle moving in the $z$-direction through a gas of scattering centers. As it moves, it suffers many small angle collisions given by $v_\perp$ which can be decomposed into random variables $\Delta v_x$ and $\Delta v_y$. These latter have just the properties of our random variable $\Delta x_i$ above. For one collision, with a given impact parameter $p$ (Figure 2.2), we have from (2-30)

$$\langle v_\perp^2 \rangle = \langle (\Delta v_x)^2 \rangle + \langle (\Delta v_y)^2 \rangle = \frac{v_0^2 p_0^2}{p^2}. \quad (2-34)$$

Since $\Delta v_x$ must have the same statistical properties as $\Delta v_y$, we must have

$$\langle (\Delta v_x)^2 \rangle = \langle (\Delta v_y)^2 \rangle = \frac{1}{2} \frac{v_0^2 p_0^2}{p^2}. \quad (2-35)$$

Then by (2-33) we have, for the total $x$ velocity $\nabla v_x^{tot}$,

$$\langle (\Delta v_x^{tot})^2 \rangle = N \langle (\Delta v_x)^2 \rangle = \frac{N v_0^2 p_0^2}{2 p^2}. \quad (2-36)$$

Since we are considering a particle moving through a gas of scattering centers, it is more useful for our purposes to have the time derivative of (2-36), where on the right we shall have $dN/dt = 2\pi p dp n_0 v_0$ as the number of scattering centers. With impact parameter between $p$ and $p + dp$, which our incident particle encounters per unit time, the time derivative of (2-36) is then
Figure 2.2 The incident particle is located at the origin and is traveling into the paper. It makes simultaneous small angle collisions with all of the scattering centers randomly distributed with impact parameters between \( p \) and \( p + dp \).

\[
\frac{d}{dt} \left( \langle (\Delta v_x^{tot})^2 \rangle \right) = \pi n_0 v_0^2 p_0^2 \frac{dp}{p}. 
\] (2-37)

We have calculated (2-37) for only one set of impact parameters between \( p \) and \( p + dp \). The same logic that led to (2-33) also allows us to sum (integrate) the right side of (2-37) over all impact parameters to obtain a total change in mean square velocity in the \( \hat{x} \)-direction and the total change in mean square velocity in the \( \hat{y} \)-direction to obtain a total mean square perpendicular velocity \( \langle (\nabla v_{\parallel}^{tot})^2 \rangle \). With this final factor of two we have

\[
\frac{d}{dt} \left( \langle (\Delta v_{\perp}^{tot})^2 \rangle \right) = 2\pi n_0 v_0^2 p_0^2 \int_{p_{min}}^{p_{max}} \frac{dp}{p}. 
\] (2-38)

What should we use for \( p_{max} \) and \( p_{min} \)? Recall that our derivation of the scattering angle \( v_{\perp}/v_0 \) in (2-30) uses the Coulomb force law. However, we know from Section 2.2.1 that the true force law is modified by Debye shielding and is essentially negligible at distances (impact parameters) much greater than a Debye length. Thus, \( p_{max} = \lambda_D \). In the case of \( p_{min} \), we use the fact that our scattering formula (2-30) is not valid for impact parameters \( p < |p_0| \) so replace \( p_{min} \) by \( |p_0| \). Equation (2-38) is then
\[
\frac{d}{dt} \left( (\Delta v_{\perp}^{tot})^2 \right) = 2\pi n_0 v_0^3 p_0^2 \ln \left( \frac{\lambda_D}{|p_0|} \right). \tag{2-39}
\]

Since the logarithm is such a slowly varying function of its argument, it will suffice to make a very rough evaluation of \( \lambda_D/p_0 \). In the definition of \( p_0 \) in (2-29) we take \( q = -e \), \( q_0 = +e \), \( m = m_e \), and for this rough calculation replace \( v_0 \) by the electron thermal speed \( v_e \) to obtain

\[
\frac{\lambda_D}{|p_0|} \approx \frac{\lambda_D m_e v_e^2}{2e^2} \approx \frac{m_e \lambda_D^3 \omega_e^2}{2e^2} \approx 2\pi n_0 \lambda_D^3 = 2\pi \Lambda. \tag{2-40}
\]

Where we have ignored the difference between \( \lambda_D \) and \( \lambda_e \). Dropping the small factor 2\( \pi \), compared with the large plasma parameter \( \Lambda \), and using the definition (2-29) of \( p_0 \), we find that (2-39) becomes

\[
\frac{d}{dt} \left( (\Delta v_{\perp}^{tot})^2 \right) = \frac{8\pi n_0 e^4}{m_e^2 v_0^3} \ln \Lambda. \tag{2-41}
\]

A reasonable definition for the scattering time due to small angle collisions is the time it takes \( (\Delta v_{\perp}^{tot})^2 \) to equal \( v_0^2 \) according to (2-41). The inverse of this time is the collision frequency \( \nu_c \) due to small-angle collisions,

\[
\nu_c = \frac{8\pi n_0 e^4 \ln \Lambda}{m_e^2 v_0^3}, \tag{2-42}
\]

note again the inverse cube dependence on the velocity \( v_0 \). One important aspect of \( \nu_c \) is that it is a factor 2\( \ln \Lambda \) larger than the collision frequency \( \nu_L \) for large angle collisions given by (2-31). This is a substantial factor in plasma (\( \ln \Lambda = 14 \) if \( \Lambda = 10^6 \)). Thus, the deflection of a charged particle in plasma is predominantly due to the many random small angle collisions that it suffers, rather than the rare large angle collisions.

Throughout one’s study of plasma physics, it is useful to identify each phenomenon as a collective effect or as a single particle effect. The oscillation of the plasma slab in
section 2.2.3, characterized by the plasma frequency $\omega_e$, is a collective effect involving many particles acting simultaneously to produce a large electric field. The collisional deflection of a particle, represented by the collision frequency $v_c$ in (2-42), is a single particle effect caused by many collisions with individual particles that do not act cooperatively.

Debye shielding described in Section 2.2.1 is a collective effect and not a single particle effect. It is instructive to calculate the ratio of $v_c$ to $\omega_e$, which is, taking a typical speed $v_o$ in (2-42),

$$\frac{v_c}{\omega_p} \approx \frac{8\pi n_0 e^4 \ln \Lambda}{m_e^2 v_0^3 \omega_e} = \frac{\ln \Lambda}{2\pi n_0 \Lambda_e^3} = \frac{\ln \Lambda}{2\pi \Lambda_e^3}.$$  \hspace{1cm} (2-43)

By crudely dropping the factor $\ln \Lambda / 2\pi$ and replacing $\Lambda_e$ by $\Lambda$, we have the easily remembered but very approximate expression

$$\frac{v_c}{\omega_p} \approx \frac{1}{\Lambda}.$$  \hspace{1cm} (2-44)

Thus, the collision frequency in plasma is very much smaller than the plasma frequency. In this respect, single particle effects are less important than collective effects. A wave with frequency near $\omega_e$ will oscillate many times before being substantially damped because of collisions [7].
2.3 Plasma Kinetic Theory (Klimontovich Equation)

2.3.1 Introduction

In this section, we begin a study of the basic equations of plasma physics. The word “kinetic” means “pertaining to motion” so that plasma kinetic theory is the theory of plasma taking into account the motions of all the particles. This can be done in an exact way, using the Klimontovich or Liouville equation. However, we are usually not interested in the exact motion of all of the particles in plasma, but rather in certain average or approximate characteristics. Thus, the greatest usefulness of the exact Klimontovich and Liouville equations is as starting points for the derivation of approximate equations that describe the average properties of plasma.

In classic plasma physics, we think of the particles as point particles, each with a given charge and mass. Suppose we have a gas consisting of only one particle. This particle has an orbit $X_1(t)$ in three-dimensional configuration space $x$. The orbit $X_1(t)$ is the set of positions $x$ occupied by the particle at successive times $t$. Likewise, the particle has an orbit $V_1(t)$ in three-dimensional velocity space $v$. We combine three-dimensional configuration space $x$ and three-dimensional velocity space $v$ into six-dimensional phase space $(x, v)$. The density of one particle in this phase space is

$$N(x, v, t) = \delta[x - X_1(t)]\delta[v - V_1(t)].$$

(2-45)

Where $\delta[x - X_1] \equiv \delta(x - X_1)\delta(y - Y_1)\delta(z - Z_1)$, and $\delta$ shows the Direct delta function. Note that $X_1, V_1$ are the Lagrangian coordinates of the particle itself, whereas $x, v$ are the Eulerian coordinates of the phase space. At any time $t$, the density of particles integrated over all phase space must yield the total number of particles in the system.
Suppose there is a system with two point particles, with respective orbits \([\mathbf{X}_1(t), \mathbf{V}_1(t)]\) and \([\mathbf{X}_2(t), \mathbf{V}_2(t)]\) in phase space \((\mathbf{x}, \mathbf{v})\). By analogy to (2-45), the particle density is

\[
N(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{2} \delta[\mathbf{x} - \mathbf{X}_i(t)]\delta[\mathbf{v} - \mathbf{V}_i(t)].
\]  

(2-46)

Now suppose that a system contains two species of particles, electrons and ions, and each species has \(N_0\) particles. Then the density \(N_s\) of species \(s\) is

\[
N_s(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N_0} \delta[\mathbf{x} - \mathbf{X}_i(t)]\delta[\mathbf{v} - \mathbf{V}_i(t)].
\]  

(2-47)

and the total density \(N\) is

\[
N(\mathbf{x}, \mathbf{v}, t) = \sum_{s, i} N_s(\mathbf{x}, \mathbf{v}, t).
\]  

(2-48)

If we know the exact positions and velocities of the particles at one time, then we know them at all later times. This can be seen as follows. The position \(\mathbf{X}_i(t)\) of particle \(i\) satisfies the equation

\[
\dot{\mathbf{X}}_i(t) = \mathbf{V}_i(t).
\]  

(2-49)

Where, an overdot means a time derivative. Likewise, the velocity \(\mathbf{V}_i(t)\) of particle \(i\) satisfies the Lorentz force equation.

\[
m_s \ddot{\mathbf{V}}_i(t) = q_s \mathbf{E}^m[\mathbf{X}_i(t), t] + \frac{q_s}{c} \mathbf{V}_i(t) \times \mathbf{B}^m[\mathbf{X}_i(t), t].
\]  

(2-50)

Where, the superscript \(m\) indicates that the electric and magnetic fields are the microscopic fields self-consistently produced by the point particles themselves, together with externally applied fields.

### 2.3.2 Klimontovich Equation

An exact equation for the evolution of plasma is obtained by taking the time derivative of the density \(N_s\). From (2.47), this is
\[
\frac{\partial N_s(x, v, t)}{\partial t} = \sum_{i=1}^{N_0} \dot{X}_i \cdot \nabla_x \delta[x - X_i(t)] \delta[v - V_i(t)] \\
- \sum_{i=1}^{N_0} \dot{V}_i \cdot \nabla_v \delta[x - X_i(t)] \delta[v - V_i(t)].
\]

(2-51)

Where we have the relations

\[
\frac{\partial}{\partial a} f(a - b) = -\frac{\partial}{\partial b} f(a - b)
\]

and

\[
\frac{\partial}{\partial t} f[g(t)] = \frac{df}{dg} \dot{g}
\]

and where \( \nabla_x \equiv (\partial_x, \partial_y, \partial_z) \) and \( \nabla_v \equiv (\partial_{v_x}, \partial_{v_y}, \partial_{v_z}) \). Using (2-49) and (2-50), we can write \( \dot{X}_i \) and \( \dot{V}_i \) in terms of \( V_i \) and the fields \( E^m \) and \( B^m \), whereupon (2-51) becomes

\[
\frac{\partial N_s(x, v, t)}{\partial t} = - \sum_{i=1}^{N_0} V_i \cdot \nabla_x \delta[x - X_i] \delta[v - V_i] \\
- \sum_{i=1}^{N_0} \left\{ \frac{q_s}{m_s} E^m [X_i(t), t] + \frac{q_s}{m_sc} V_i \times B^m [X_i(t), t] \right\} \cdot \nabla_v \delta[x - X_i] \delta[v - V_i].
\]

(2-52)

The effect of collisions will appear in the final equation as a result of the discrete-nature of the plasma, and important property of the Dirac delta function is

\[ a\delta(a - b) = b\delta(a - b) \]

This relation allows us to replace \( V_i(t) \) with \( v \), and \( X_i(t) \) with \( x \), on the right of (2-52) (but not in the arguments of the delta functions) so that (2-52) becomes

\[
\frac{\partial N_s(x, v, t)}{\partial t} = -v \cdot \nabla_x \sum_{i=1}^{N_0} \delta[x - X_i] \delta[v - V_i] \\
- \left[ \frac{q_s}{m_s} E^m (x, t) + \frac{q_s}{m_sc} v \times B^m (x, t) \right] \cdot \nabla_v \sum_{i=1}^{N_0} \delta[x - X_i] \delta[v - V_i].
\]

(2-53)

But the two summations on the right of (2-53) are just the density (2-47); therefore
\[ \frac{\partial N_s(x,v,t)}{\partial t} + v \cdot \nabla_x N_s + \frac{q_s}{m_s} \left( E^m + \frac{v}{c} \times B^m \right) \cdot \nabla_v N_s = 0 \]  

(2-54)

This is the exact Klimontovich equation [8]. The Klimontovich equation, together with Maxwell’s equations, constitutes an exact description of plasma. Given the initial positions and velocities of the particles, the initial densities \( N_e(x,v,t = 0) \) and \( N_i(x,v,t = 0) \) are given exactly by (2-47). The initial fields are then chosen to be consistent with Maxwell’s equations. With these initial conditions the problem is completely deterministic, and the densities and fields are exactly determined for all time.

In practice, we never carry out this procedure. The Klimontovich equation contains every one of the exact single particle orbits. This is far more information than we want or need. What we really want is information about certain average properties of the plasma. We do not really care about all of the individual electromagnetic fields contributed by the individual charges. What we do care about is the average long-range electric field, which might exist over many thousands or millions of inter-particle spacing. The usefulness of the Klimontovich equation comes from its role as a starting point in the derivation of equations that describe the average properties of plasma.

The Klimontovich equation can be thought of as expressing the incompressibility of the “substance” \( N_s(x,v,t) \) as it moves about in the \( (x,v) \) phase space. This can be seen as follows. Imagine a hypothetical particle charge \( q_s \), mass \( m_s \), which at time \( t \) finds itself at the position \( (x,v) \). This hypothetical particle has an orbit in phase space determined by the fields in the system. Imagine taking a time derivative of any quantity along this orbit (such a time derivative is called a convective derivative). This derivative must include the time variation produced by the changing position in \( (x,v) \) space as well as the explicit time variation of the quantity. Thus, it must be given by
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\big|_{\text{orbit}} \cdot \nabla_x + \frac{dv}{dt}\big|_{\text{orbit}} \cdot \nabla_v.\tag{2-55}
\]

Where by \( \frac{dx}{dt}\big|_{\text{orbit}} \) we mean the change in position \( x \) of the hypothetical particle with time; likewise for \( \frac{dv}{dt}\big|_{\text{orbit}} \). But for our hypothetical particle at position \((x, v)\) in phase space we know that
\[
\frac{dx}{dt}\big|_{\text{orbit}} = v, \tag{2-56}
\]
and
\[
\frac{dv}{dt}\big|_{\text{orbit}} = \frac{q_s}{m_s} \left[ E^m(x, t) + \frac{v}{c} \times B^m(x, t) \right]. \tag{2-57}
\]
Thus,
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla_x + \frac{q_s}{m_s} \left[ E^m(x, t) + \frac{v}{c} \times B^m(x, t) \right] \cdot \nabla_v, \tag{2-58}
\]
and the Klimontovich equation (2-54) simply says
\[
\frac{D}{Dt} N_s(x, v, t) = 0. \tag{2-59}
\]

The density of particles of species \( s \) is a constant in time, as measured along the orbit of a hypothetical particle of species \( s \). This is true whether we are moving along the orbit of an actual, in which case the density is infinite, or whether we are moving along a hypothetical orbit that is not occupied by an actual particle, in which case the density is zero. Note that the density is only constant as measured along orbits of hypothetical particles; in \((x, v)\) space at a given time it is not constant but is zero or infinite.

There is yet a third way to think of the Klimontovich equation. Any fluid in which the fluid density \( f(r, t) \) is neither created nor destroyed satisfies a continuity equation
\[
\partial_t f(r, t) + \nabla_r \cdot (fV) = 0, \tag{2-60}
\]
where $\nabla r$ is the divergence vector in the phase space under consideration, and $\mathbf{V}$ is a vector that gives the time rate of change of a fluid element at a point in phase space. In the present case $\nabla r = (\nabla_x, \nabla v)$ and $\mathbf{V} = (d\mathbf{X}/dt|_{\text{orbit}}, d\mathbf{V}/dt|_{\text{orbit}})$. Since the particle density is neither created nor destroyed, it must satisfy a continuity equation of the form

$$\partial_t N_s(x, v, t) + \nabla_x \cdot (v N_s) + \nabla_v \cdot \left\{ \frac{q_s}{m_s} \left[ \mathbf{E}^m(x, t) + \frac{v}{c} \times \mathbf{B}^m(x, t) \right] N_s \right\} = 0. \quad (2-61)$$

It can be proved that the continuity Equation (2-61) is equivalent to the Klimontovich equation [8], [9]. The effect of collisions is hidden in the third term of the equation.

### 2.3.3 Plasma Kinetic Equation

Although the Klimontovich equation is exact, we are really not interested in its exact solution. These would contain all of the particle orbits, and would thus be far too detailed for practical purposes. What we really would like to know are the average properties of plasma. The Klimontovich equation tells us whether or not a particle with infinite density is to be found at a given point $(x, v)$ in phase space. What we really want to know is how many particles are likely to be found in a small volume $\Delta x \Delta v$ of phase space whose center is at $(x, v)$. Thus, we really are not interested in the spikey function $N_s(x, v, t)$, but rather in the smooth function

$$f_s(x, v, t) \equiv \langle N_s(x, v, t) \rangle. \quad (2-62)$$

The most rigorous way to interpret $\langle \rangle$ is as an ensemble average over an infinite number of realizations of the plasma, prepared according to some prescription. For example, we could prepare an ensemble of equal temperature plasmas, each in thermal equilibrium, and each with a test charge $q_T$ at the origin of configuration space. The
resulting \( f_e \) and \( f_i \) would then be consistent with the discussion of Debye shielding in section 2.2.1.

There is another useful interpretation of the distribution function \( f_s(x, v, t) \), the number of particles of species \( s \) per unit configuration space per unit velocity space. Suppose we are interested in long range electric and magnetic fields that extend over distances much larger than a Debye length. Then we can imagine a box, centered around the point \( x \) in configuration space, of a size much greater than a mean inter-particle spacing, but much smaller than a Debye length. We can now count the number of particles of species \( s \) in the box at time \( t \) with velocities in the range \( v \) to \( v+\Delta v \), divide by (the size of the box multiplied by \( \Delta v_x \Delta v_y \Delta v_z \)), and call the result \( f_s(x, v, t) \). This number will of course fluctuate with time but, if there are very many particles in the box, the fluctuations will be tiny and the \( f_s(x, v, t) \) obtained in this manner will agree very well with that obtained in the more rigorous ensemble averaging procedure.

An equation for the time evolution of the distribution function \( f_s(x, v, t) \) can be obtained from the Klimontovich Equation (2-54) by ensemble averaging. We define \( \delta N_s, \delta E, \) and \( \delta B \) by

\[
N_s(x, v, t) = f_s(x, v, t) + \delta N_s(x, v, t),
\]
\[
E^m(x, v, t) = E(x, v, t) + \delta E(x, v, t), \quad \text{and} \quad \tag{2-63}
\]
\[
B^m(x, v, t) = B(x, v, t) + \delta B(x, v, t).
\]

where \( B \equiv \langle B^m \rangle, E \equiv \langle E^m \rangle, \) and \( \langle \delta N_s \rangle = \langle \delta E \rangle = \langle \delta B \rangle = 0. \) Inserting these definitions into (2-54) and ensemble averaging, we obtain

\[
\frac{\partial f_s(x, v, t)}{\partial t} + \mathbf{v} \cdot \nabla_x f_s + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \nabla_v f_s = -\frac{q_s}{m_s} \left( \left( \delta E + \frac{\mathbf{v}}{c} \times \delta \mathbf{B} \right) \cdot \nabla_v \delta N_s \right). \quad (2-64)
\]
Equation (2-64) is the exact form of the plasma kinetic equation. The left side of (2-64) consists only of terms that vary smoothly in \((x, v)\) space. The right side is the ensemble average of the products of very spikey quantities like \(\delta E = E^m - \langle E^m \rangle\) and \(\delta N_s\). Thus, the left side of (2-64) contains terms that are insensitive to the discrete-particle nature of the plasma, while the right side of (2-64) is very sensitive to the discrete-particle nature of the plasma. But the discrete-particle nature of plasma is what gives rise to collisional effects, so that the left side of (2-64) contains smoothly varying functions representing collective effects, while the right side represents the collisional effects. We have seen in section 2.2.4 that the ratio of the importance of collisional effects to the importance of collective effects is sometimes given by \(1/\Lambda\), which is a very small number. We might guess that for many phenomena in plasma, the right side of (2-64) has a size \(1/\Lambda\) compared to each of the terms on the left side [8]; thus the right side can be neglected for the study of such phenomena.

This important point can be illustrated by a hypothetical exercise. Imagine that we break each electron into an infinite number of pieces, so that \(n_o \rightarrow \infty, m_e \rightarrow 0,\) and \(e \rightarrow 0,\) while \(n_o e = \text{constant}, \) and \(e/m_e = \text{constant},\) and \(v_e = \text{constant}.\) Then, we can show for a hypothetical case where \(\omega_e = \text{constant, }\lambda_e = \text{constant, then } T_e \rightarrow 0\) and \(\Lambda_e \rightarrow 0.\) Any volume, no matter how small, would contain an infinite number of point particles, each represented by a delta function with infinitesimal charge. Statistical mechanics tells us that the relative fluctuations in such plasma would vanish, since the fluctuation in the number of particles \(N_0\) in a certain volume is proportional to the square root of that number. Thus, on the right side of (2-64) we have \(\delta N_s \sim N_0^{1/2} \sim \Lambda_e^{1/2},\) and \(\delta E\) and \(\delta B\) which are produced by \(\delta N_s\) behaving like (from Poisson’s equation) \(\sim e \delta N_s \sim N_0^{-1} N_0^{1/2}\)
\( \sim N_0^{-1/2} \sim \Lambda_e^{-1/2} \), so that the right side becomes constant. On the left, however, each term becomes infinite as \( f_s \to \infty \). Thus, the relative importance of the right side vanishes \( \sim N_0^{-1} \sim \Lambda_e^{-1} \), and we have

\[
\frac{\partial f_s(x,v,t)}{\partial t} + \mathbf{v} \cdot \nabla_x f_s + \frac{q_s}{m_s} \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \nabla_v f_s = 0. \tag{2-65}
\]

Which is the Vlasov equation (sometimes referred as the collisionless Boltzmann equation). This approximate equation, which neglects collisional effects, is often called the most important equation in plasma physics.

The fields \( \mathbf{E} \) and \( \mathbf{B} \) of (2-65) are the ensemble averaged fields of (2-63). They must satisfy the ensemble averaged versions of Maxwell’s equations.

\[
\nabla \cdot \mathbf{E}(x,t) = 4\pi \rho,
\]

\[
\nabla \cdot \mathbf{B}(x,t) = 0,
\]

\[
\nabla \times \mathbf{E}(x,t) = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
\]

\[
\nabla \times \mathbf{B}(x,t) = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t},
\]

\[
\rho(x,t) \equiv \langle \rho^m \rangle = \sum_{e,i} q_s \int dv f_s(x,v,t), \quad \text{and}
\]

\[
\mathbf{J}(x,t) \equiv \langle \mathbf{J}^m \rangle = \sum_{e,i} q_s \int dv \mathbf{v} f_s(x,v,t). \tag{2-66}
\]

### 2.4 Fluid Equations

#### 2.4.1 Introduction

There are many phenomena in plasma physics that can be studied by thinking of the plasma as two interpenetrating fluids, an electron fluid and an ion fluid. In this approach, it is not necessary to consider the fact that each species consists of particles with different
velocities. The advantage of this approach is its simplicity; it leads to equations in three spatial dimensions and time rather than in the seven-dimensional phase space of Vlasov theory. The disadvantage of this approach is that it misses velocity-dependent effects such as Landau damping as a result of approximations. In this section, we introduce the fluid equations heuristically.

The first equation of fluid theory is the continuity equation, which expresses the fact that the fluid is not being created or destroyed, so that the only way that the fluid density \( n_s(\mathbf{x}, t) \) of fluid species \( s \) can change at a point \( \mathbf{x} = (x, y, z) \) is by having a net amount of fluid enter or leave a small spatial volume including that point. The density \( n_s \) is the number of particles of species \( s \) per unit volume. To every element of fluid there corresponds a velocity vector \( \mathbf{V}_s(\mathbf{x}, t) \) that gives the velocity of the fluid element at the point \( \mathbf{x} \) at time \( t \). Mathematically, the continuity equation for fluid species \( s \) is

\[
\frac{\partial n_s(\mathbf{x}, t)}{\partial t} + \nabla \cdot (n_s \mathbf{V}_s) = 0.
\]  

(2-67)

where \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \) is the usual gradient operator in three-dimensional configuration space. The second equation of fluid theory is the force equation, which is simply Newton’s second law of motion for a fluid. This can be written for fluid species \( s \) as

\[
n_s m_s \dot{\mathbf{V}}_s(\mathbf{x}, t) = \mathbf{F}_s(\mathbf{x}, t).
\]  

(2-68)

Where \( \mathbf{F}_s(\mathbf{x}, t) \) is the force per unit volume acting on the fluid element at position \( \mathbf{x} \) at time \( t \). The time derivative on the left of Newton’s law refers to the fluid element as an entity and therefore must be taken along the orbit of the fluid element. Thus,

\[
\dot{\mathbf{V}}_s(\mathbf{x}, t) = \partial_t \mathbf{V}_s + \left( \frac{dx}{dt} \right) \mathbf{V}_s = \partial_t \mathbf{V}_s + (\mathbf{V}_s \cdot \nabla) \mathbf{V}_s.
\]  

(2-69)
On the right side of (2-68) are all of the forces that act on a fluid element. One such force is the pressure gradient force. A fluid of charged particles has a pressure \( P_s(x, t) = n_s(x, t)T_s(x, t) \) and an associated force per unit volume \(-\nabla P_s\). Another force is the Lorentz force per unit volume, \( q_s n_s(x, t)E(x, t) + (q_s/c)n_s(x, t)V_s(x, t) \times B(x, t)\). If these forces are considered, (2-68) becomes

\[
\begin{align*}
n_s m_s \partial_t V_s + n_s m_s V_s \cdot \nabla V_s &= -\nabla P_s + q_s n_s E + \frac{q_s}{c} n_s V_s \times B, \\
\partial_t V_s + V_s \cdot \nabla V_s &= -\frac{1}{n_s m_s} \nabla P_s + \frac{q_s}{m_s} E + \frac{q_s m_s c}{m_s} V_s \times B.
\end{align*}
\]

Equation (2-71) can be thought of as the force equation per particle. The fields \( E(x, t) \) and \( B(x, t) \) are the macroscopic fields (those which would be measured by a probe). With the given fluid quantities, the total charge density \( \rho \) is defined by

\[
\rho(x, t) = \sum_s q_s n_s(x, t),
\]

while the total current density \( J \) is defined by

\[
J(x, t) = \sum_s q_s n_s(x, t)V_s(x, t).
\]

When combined with Maxwell’s equations, the fluid equation provides a complete, but approximate, description of plasma physics. A more careful development of the fluid equations from the Vlasov equation is provided in the next section.

---

2.4.2 The derivation of the fluid equations from the Vlasov equation

Except for the neglect of collisions, the Vlasov equation is an exact description of plasma. By taking velocity moments of the Vlasov equation in seven-dimensional \((x, v, t)\) space, an infinite hierarch of equations in four-dimensional \((x, t)\) space can be derived. When an appropriate truncation of this infinite hierarchy is carried out, the standard two-fluid theory of plasma physics is obtained. This procedure is reminiscent of
the truncation of Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy that led to the plasma kinetic equation and hence to the Vlasov equation [11], [12]. The Vlasov equation (2-65) is
\[ \partial_t f_s(x, v, t) + v \cdot \nabla_x f_s + \frac{q_s}{m_s} \left( E + \frac{v}{c} \times B \right) \cdot \nabla_v f_s = 0. \] (2-74)
we use the normalization
\[ n_s(x, t) = \int dv f_s(x, v, t), \] (2-75)
and note that the fluid velocity
\[ V_s(x, t) = \frac{1}{n_s} \int dv v f_s(x, v, t). \] (2-76)
The first fluid equation (the continuity equation) is obtained by integrating (2-74) over all velocity space (we first multiply by “unity”). The first term of (2-74) yields
\[ \frac{\partial n_s(x, t)}{\partial t} + \nabla_x \cdot (n_s V_s) = 0. \] (2-78)
To prove above discussion we use (2-74). The force equation is obtained by multiplying (2-74) by \(v\) and integrating over all velocity space. This yields,
\[ \frac{\partial}{\partial t} \int dv v f_s + \int dv vv \cdot \nabla_x f_s + \frac{q_s}{m_s} \int dv v \left[ \left( E + \frac{v}{c} \times B \right) \cdot \nabla_v f_s \right] = 0. \] (2-79)
In the first term of (2-79), we have \(\frac{\partial (n_v V_s)}{\partial t}\) by (2-76). In the second term, we perform the manipulation \(v v \cdot \nabla_x f_s = v \cdot \nabla_x (v f_s) = \nabla_x \cdot (v f_s)\). Since \(f_s(x, v, t)\) is a probability distribution, the ensemble average of any quantity is
\begin{equation}
\langle g \rangle = \frac{\int dv g f_s}{\int dv f_s} = \frac{1}{n_s} \int dv g f_s,
\end{equation}

thus, the second term is

\begin{equation}
\nabla_x \cdot \int dv vv f_s = \nabla_x \cdot (n_s \langle vv \rangle).
\end{equation}

The third term is easily evaluated by integration by parts, yielding \(- (q_s n_s/m_s)E\). In the fourth term of (2-79), it is useful to move \( \nabla v \) to the left, obtaining \( \nabla_v \cdot [(v \times B)f_s] \); and integrating by parts then yields

\begin{equation}
\text{Fourth term}= -\frac{q_s}{m_sc} \int dv (v \times B) f_s = -\frac{q_s}{m_sc} n_s (V_s \times B),
\end{equation}

where we have evaluated each component in (2-79) using (2-76). Combining all terms, (2-79) becomes

\begin{equation}
\partial_t (n_s V_s) + \nabla \cdot (n_s \langle vv \rangle) = \frac{q_s}{m_s} n_s \left( E + \frac{V_s}{c} \times B \right).
\end{equation}

which is the fluid force equation for species \( s \). Multiplying through by the mass \( m_s \) [13], we see that each term has units of (force/volume). Equation (2-83) is also called the momentum equation, since it determines the time rate of change of momentum per unit volume. By simplifying equation (2-83) and considering the effect of collisions and \( B = 0 \) the Drude-Lorentz equation can be derived.

### 2.5 The Effect of AC Electric Fields

The foregoing aspects of particle motion do not discuss the exact effect of ac electric fields in depth. Such fields are extremely commonplace in plasmas. Whenever electromagnetic radiation interacts with plasma, plasma waves or instabilities appear and whenever the effects of neighboring charges must be considered, an ac or a fluctuating field is often the result.
In the case of applying a dc magnetic field, we shall assume that the dc magnetic field is sufficiently strong to outweigh the effect of any ac magnetic field associated with the fluctuating electric field. If the ac magnetic field is to be included, however, it may be obtained in terms of the electric field from Maxwell’s equations.

We will now investigate the behavior of charged particles in the presence of a dc magnetic field and ac electric field. Figure 2.3 shows the coordinate geometry used for this problem.

![Figure 2.3 Coordinate system for charged particle motion in an ac electric field.](image)

Let us consider that the charged particle is an electron. We shall break up the electric field into two components, $E_\parallel$ and $E_\perp$. Their directions are related to the dc magnetic field direction that is shown here as being parallel to the $z$ axis. The dc magnetic field is assumed to be so large that the induced ac magnetic fields, from the ac electric field, can be neglected. The nonrelativistic equation of motion of the electron is then

$$m_e \frac{d\mathbf{V}}{dt} = q_e E_\parallel + q_e E_\perp + q_e (\mathbf{V} \times \mathbf{B}).$$

(2-84)
We can solve this equation directly for its complete solution, if initial conditions on velocity and electric field are known. Let us now assume, first, that \( \mathbf{V} \), the velocity of the electron, may be broken up into two components, as shown in equation (2-85)

\[
\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2.
\]  

(2-85)

We shall let \( \mathbf{V}_2 \) be the motion of the electron in the presence of the dc magnetic field alone. It must therefore be a solution to the following equation,

\[
m_e \frac{d\mathbf{V}_2}{dt} = q_e (\mathbf{V}_2 \times \mathbf{B}).
\]  

(2-86)

\( \mathbf{V}_2 \) will be zero, if there is no random thermal motion. Equation (2-86) provides for no mechanism to make \( \mathbf{V}_2 \) other than zero, except for initial conditions. We may express \( \mathbf{V}_2 \) in component directions as

\[
\mathbf{V}_2 = \mathbf{V}_{2\parallel} - (\mathbf{\omega}_c \times \mathbf{r}_g).
\]  

(2-87)

where \( \mathbf{r}_g \) is a vector pointing to the particle from the center of the orbit whose magnitude is the radius of gyration. We shall define \( \mathbf{\omega}_c \) for the electrons and the ions to be \( q \mathbf{B}/m \).
The sign of the charge is carried with \( \mathbf{\omega}_c \) by \( q \). For ions, \( \mathbf{\omega}_c \) points in the same direction as \( \mathbf{B} \).

On the other hand, \( \mathbf{V}_1 \) is the velocity of the particle when the electric field is present. It is a superposition of the velocity upon the one described by (2-86), that gives the net particle motion, therefore, it must be such that \( \mathbf{V}_1 \) is zero when \( \mathbf{E} \) is zero. This separation, defined in (2-85), is known as the guiding center approximation [14-16].

We may restate this approximation. It is essentially the assumption that the complete motion of the particle may be made by superimposing the gyration at the cyclotron frequency \( \mathbf{\omega}_c \) (speed \( \mathbf{V}_2 \)) of the particle over the motion of the center of the orbit \( \mathbf{V}_1 \). We may compute the motion of the guiding center by use of the Lorentz force equation (2-
50) with the addition of forces dependent upon the non-uniformities in the electric and magnetic field if they vary across the orbit diameter. If the particle started up from zero velocity, the guiding center solution is exactly the complete solution, since the guiding center and the actual position of the particle coincide. As the particle gains energy and begins to spiral around the lines of force, the approximation may become less and less valid, depending upon non-uniformities in the electric and magnetic fields that may appear as the orbit diameter increases. The approximation is always valid if there is no random orbital motion, since the momentum equation may be solved exactly. Recall again that it is also always valid if there are no non-uniformities in the electric and magnetic fields, regardless of the particle’s velocities.

The drift velocities previously defined are the drifts of the guiding centers in fields that do not change in time. For the following development, we compute the velocity of the guiding center when it is subjected to a time-changing electric field. It will be seen that “orbital” motion of the guiding center does develop, but again, this motion must be superimposed on the original “random” orbital motion before the complete path of the particle is known. As the frequency of the electric field becomes comparable to the cyclotron frequency, the guiding center approximation becomes less valid although we may still use an expansion for the particle’s motion like equation (2-85). In this case, \( V_1 \) and \( V_2 \) are velocities of the particle itself, but only \( V_1 \) is depended upon the ac electric field.

We now define a rectangular set of coordinates, shown in Figure 2.4, which can be used to describe the motion of \( V_1 \). They are superimposed over the ones of Figure 2.3.
The three coordinate vectors, are  \( \mathbf{E}_\parallel, \mathbf{E}_\perp, \) and \( \mathbf{E}_\perp \times \mathbf{c}_e \). Note that they are all mutually perpendicular, but they are not Unit vectors.

For electrons, these vectors from a right-handed set of coordinates, since

\[
\frac{\mathbf{E}_\parallel \times \mathbf{E}_\perp}{|\mathbf{E}_\parallel||\mathbf{E}_\perp|} = \frac{\mathbf{E}_\perp \times \mathbf{c}_e}{|\mathbf{E}_\perp||\mathbf{c}_e|},
\]

(2-88)

For ions the vectors must be \( \mathbf{E}_\parallel, \mathbf{E}_\perp, \) and \( - (\mathbf{E}_\perp \times \mathbf{c}_i) \) to ensure right-handedness.

We will now follow the electrons. We can determine the velocity \( \mathbf{V}_1 \), which must obey

(2-89)

\[
m_e \frac{d\mathbf{V}_1}{dt} = q_e (\mathbf{E} + \mathbf{V}_1 \times \mathbf{B}).
\]

(2-89)

---

Figure 2.4 Right-handed coordinates \( \mathbf{E}_\parallel, \mathbf{E}_\perp, \) and \( \mathbf{E}_\perp \times \mathbf{c}_e \).

\( \mathbf{V}_1 \) is a linear function of \( \mathbf{E} \) and has components in the directions \( \mathbf{E}_\parallel, \mathbf{E}_\perp, \) and \( \mathbf{E}_\perp \times \mathbf{c}_e \), defined by the new set of coordinates. Therefore

\[
\mathbf{V}_1 = a_0 \mathbf{E}_\parallel + a_0 \mathbf{E}_\perp + a_2 (\mathbf{E}_\perp \times \mathbf{c}_e).
\]

(2-90)

The coefficients \( a_0, a_1, \) and \( a_2 \) are found by solving for the steady state solution of equation (2-89) with an electric field present. The field may vary in time, but only
sinusoidally. This causes little loss in generality, since a Fourier representation of any periodic signal may be obtained. Note that \( a_0, a_1, \) and \( a_2 \) may be functions of time.

To consider the most general case, we shall let \( \mathbf{E} \) vary as the real part of \( \mathbf{E}_0 e^{-j\omega t} \). Since \( \mathbf{V}_1 \) is a linear function of \( \mathbf{E} \), it will also vary as \( e^{-j\omega t} \). There are three cases of interest:

\[
\begin{align*}
\omega &= 0; \\
\omega &\neq 0, \omega \neq \omega_c; \\
\omega &= \omega_c.
\end{align*}
\]

We have removed the subscript for the electrons, since the three cases above will also apply to the ions. Let us consider the steady state solution to (2-89). We will not consider the general solutions to (2-89) because we are interested in longtime behavior. Although there is no damping in this equation, all real plasmas exhibit some damping; and the steady state solution should be that which remains after a long time has elapsed. The general solutions, if any, will appear as solutions to (2-86) which determines \( \mathbf{V}_2 \). The result then will be the complete solution to equation (2-84), \( \mathbf{V}_1 + \mathbf{V}_2 \), if we assume the single particle model holds.

Equation (2-89) is a vector equation and it will result in a three component equation: \( \mathbf{E}_\parallel \) components, \( \mathbf{E}_\perp \) components, and \( \mathbf{E}_\perp \times \omega_{ce} \) components. We now write (2-89) with (2-90) substituted in after dividing through by \( m_e \). This is

\[
e^{-j\omega t} [a_0 \mathbf{E}_{\parallel 0} - j\omega a_0 \mathbf{E}_{\parallel 0} + a_1 \mathbf{E}_{\perp 0} - j\omega a_1 \mathbf{E}_{\perp 0} + \hat{a}_2 (\mathbf{E}_{\perp 0} \times \omega_{ce}) - j\omega a_2 (\mathbf{E}_{\perp 0} \times \omega_{ce})] = e^{-j\omega t} \left[ \frac{q_e}{m_e} \mathbf{E}_{\parallel 0} + \frac{q_e}{m_e} \mathbf{E}_{\perp 0} + \frac{q_e}{m_e} (a_0 (\mathbf{E}_{\parallel 0} \times \mathbf{B}) + a_1 (\mathbf{E}_{\perp 0} \times \mathbf{B}) \mathbf{E}) + a_2 (\mathbf{E}_{\perp 0} \times \omega_{ce}) \times \mathbf{B} \right].
\]
Note that \( \left( \frac{q_e}{m_e} \right) \mathbf{B} = \omega_{ce} \), since the sign of the charge is carried by \( q_e \), so the right-hand side of (2-91) becomes

\[
e^{-j\omega t} \left[ \frac{q_e}{m_e} \mathbf{E}_{\|0} + \frac{q_e}{m_e} \mathbf{E}_{\perp0} + a_0 (\mathbf{E}_{\|0} \times \omega_{ce}) + a_1 (\mathbf{E}_{\perp0} \times \omega_{ce}) + a_2 (\mathbf{E}_{\perp0} \times \omega_{ce}) \times \omega_{ce} \right],
\]

Using the vector diagram of Figure 1.4 we may now simplify the foregoing expression to yield

\[
e^{-j\omega t} \left[ \frac{q_e}{m_e} \mathbf{E}_{\|0} + \frac{q_e}{m_e} \mathbf{E}_{\perp0} + a_1 (\mathbf{E}_{\perp0} \times \omega_{ce}) - a_2 \omega_{ce}^2 \mathbf{E}_{\perp0} \right].
\]

Note that \( \mathbf{E}_{\|0} \times \omega_{ce} \equiv 0 \). We can obtain three component equations from (2-91):

**\( \mathbf{E}_{\|0} \) Component**

\[
\dot{a}_0 \mathbf{E}_{\|0} - j\omega a_0 \mathbf{E}_{\|0} = \frac{q_e}{m_e} \mathbf{E}_{\|0} \quad \text{or} \quad \dot{a}_0 - j\omega a_0 = \frac{q_e}{m_e}. \quad (2-92)
\]

**\( \mathbf{E}_{\perp0} \) Component**

\[
\dot{a}_1 \mathbf{E}_{\perp0} - j\omega a_1 \mathbf{E}_{\perp0} = \frac{q_e}{m_e} \mathbf{E}_{\perp0} - a_2 \omega_{ce}^2 \mathbf{E}_{\perp0}, \quad \text{or} \quad \dot{a}_1 - j\omega a_1 + a_2 \omega_{ce}^2 = \frac{q_e}{m_e}. \quad (2-93)
\]

**\( \mathbf{E}_{\perp0} \times \omega_{ce} \) Component**

\[
\dot{a}_2 (\mathbf{E}_{\perp0} \times \omega_{ce}) - j\omega a_2 (\mathbf{E}_{\perp0} \times \omega_{ce}) = a_1 (\mathbf{E}_{\perp0} \times \omega_{ce}), \quad \text{or} \quad \dot{a}_2 - j\omega a_2 = a_1. \quad (2-94)
\]

The over-dot of coefficients in (2-91) refers to differentiation of the with respect to time. We will now obtain the steady state solutions to these equations for the three ranges of \( \omega \).

Note that when considering the ions, the third vector must be \(- (\mathbf{E}_{\perp0} \times \omega_{ci})\). What this means is that the solutions for \( a_2 \) for ion motion must be the negative of what they are for the electrons. This is the only difference in the two methods of solution. It is very important to remember that \( q \), wherever it appears, must change sign including in \( \omega_{ce} \) or \( \omega_{ci} \). From now on, we will use \( \omega_c \) without an additional subscript and note the cases for the ions.
Case 1, $\omega = 0$ (dc electric field)

The solution for $a_0$, $a_1$, and $a_2$ are

$$a_0 = \frac{q}{m} t \quad a_1 = 0 \quad a_2 = \begin{cases} \frac{q}{m\omega_c^2} & \text{electrons} \\ -\frac{q}{m\omega_c^2} & \text{ions} \end{cases}$$

The motion of a particle under these conditions results in a constantly increasing velocity in the $E_\parallel$ direction, no component of motion in the $E_\perp$ direction, and a constant (drift) velocity in the $E_\perp \times \omega_c$ direction. Therefore we may use $E_\perp \times \omega_{ce}$ or $-(E_\perp \times \omega_{ci})$ interchangeably since the sign of the coefficient $a_2$ takes care of the opposite vector direction and the $E \times B$ drifts remain the same for ions and electrons. The resultant velocity $V_1$ is

$$V_1 = \frac{q}{m} t E_\parallel \pm \frac{q}{m\omega_c^2} (E_\perp \times \omega_c), \quad \text{(minus sign for ions)}$$

obtained previously by the radius of gyration arguments. There is a constantly increasing velocity in the $E_\parallel$ direction (parallel to $B$) and a constant drift velocity in the direction perpendicular to both $E_\perp$ and $B$. Its magnitude is then seen to be equal to $E_\perp / B$, which is the previous result. We can investigate the solution of Equations (2-92), (2-93), and (2-94) for case 2.

Case 2, $\omega \neq 0$, $\omega \neq \omega_c$

The results for the steady state solution are

$$a_0 = -\frac{q}{j\omega m} \quad a_1 = -\frac{j\omega q}{m} \frac{1}{(\omega_c^2-\omega^2)} \quad a_2 = \begin{cases} \frac{q}{m(\omega_c^2-\omega^2)} & \text{electron} \\ -\frac{q}{m(\omega_c^2-\omega^2)} & \text{ion} \end{cases}$$

The result is that the velocity $V_1$ becomes, for case 2,

$$V_1 = \left[ -\frac{q}{j\omega m} E_\parallel \pm \frac{j\omega q}{m} \frac{1}{(\omega_c^2-\omega^2)} \right] \frac{1}{E_\perp} \pm \frac{q}{m} \frac{1}{(\omega_c^2-\omega^2)} (E_\perp \times \omega_c) e^{-j\omega t}.$$
Again, the minus sign is used for ions.) The subscript 0 for the $E$'s is defined as the Fourier amplitude. The velocity in all coordinate directions is constant in magnitude, but varies sinusoidally in time. This means that the time average velocity is zero for each of the $V_1$ components. Converting back to trigonometric functions, by taking the real part of (2-96), we obtain

$$V_1 = \left[ \frac{q}{\omega m} E_{\parallel 0} \sin(\omega t) + \frac{q \omega}{m} \frac{1}{(\omega_c^2 - \omega^2)} E_{\perp 0} \sin(\omega t) \pm \frac{q}{m} \frac{1}{(\omega_c^2 - \omega^2)} (E_{\perp 0} \times \omega_c) \cos(\omega t) \right].$$

(2-97)

**Case 3, $\omega = \omega_c$**

Case 3, the condition when $\omega = \omega_c$, is known as electron (or ion) cyclotron resonance.

The solutions of the three equations become

$$a_0 = -\frac{q}{j\omega m}, \quad a_1 = -\frac{q}{j2\omega_c m}(1 - j\omega_c t), \quad a_2 = \begin{cases} -\frac{qt}{j2\omega_c m} & \text{electron} \\ \frac{qt}{j2\omega_c m} & \text{ion} \end{cases}$$

Here the velocity is constant in magnitude in the direction parallel to the field lines, but the magnitude increases linearly with time in the $E_{\perp 0}$ and $E_{\perp 0} \times \omega_c$ directions. The velocity $V_1$ for Case 3 would then be

$$V_1 = \left[ -\frac{q}{j\omega_c m} E_{\parallel 0} - \frac{q}{j2\omega_c m}(1 - j\omega_c t)E_{\perp 0} \mp \frac{qt}{j2\omega_c m}(E_{\perp 0} \times \omega_c) \right] e^{-j\omega t},$$

(2-98)

The plus sign used for ions. Converting to trigonometric form, we get

$$V_1 = -\frac{q}{\omega_c m} E_{\parallel 0} \sin(\omega_c t) + \frac{q}{2\omega_c m} E_{\perp 0} \sin(\omega_c t)$$

$$+ \frac{q\omega_c t}{2\omega_c m} E_{\perp 0} \cos(\omega_c t) \pm \frac{q}{2\omega_c m} (E_{\perp 0} \times \omega_c) \sin(\omega_c t).$$

(2-99)

The minus sign is used for ions. The velocity in the plane perpendicular to $B$ will increase indefinitely with time as long as, the nonrelativistic, non-collisional, single-particle
model holds. Obviously, in a laboratory plasma, there must exist some limitation on the behavior of the plasma since the plasma particles eventually become relativistic and go off the resonance condition, collide with the vacuum chamber walls, travel to a region where the magnetic field changes, make collisions with other particles, or are subjected to a combination of all of these effects [17-20]. Also, if $\mathbf{V}_2$ is directed oppositely to $\mathbf{V}_1$, there will result a net deceleration of the particle, rather than an acceleration.

Note that we have now established a relationship between $\mathbf{V}_1$ and the electric field $\mathbf{E}$, in these three cases. Here we just consider the $\mathbf{E}$ drift as a simplified model while for plasma there are other types of drifts like diamagnetic drift. The ratio of $\mathbf{V}_1$ to $\mathbf{E}$ has been defined as the mobility. Therefore, in the plane perpendicular to the magnetic field, a velocity in one coordinate direction may be obtained from an electric field in another direction, as shown by the nonzero coefficients of the $\mathbf{E}_\perp \times \omega_c$ terms in Equations (2-95), (2-97), and (2-99). The result must be that the mobility in general is a tensor. We could develop this by expressing $\mathbf{E}$ in terms of its components in rectangular coordinates. If $\mathbf{B}$ is parallel to the $z$ axis, then the representation for $\mathbf{E}$ in rectangular coordinates is

$$\mathbf{E}_\parallel = E_x \hat{a}_x + E_y \hat{a}_y \quad \text{and} \quad \omega_c = \omega_c \hat{a}_z$$

The expression for $\mathbf{V}_1$ is then

$$\mathbf{V}_1 = a_0 E_z \hat{a}_z + a_1 (E_x \hat{a}_x + E_y \hat{a}_y) \pm a_2 [(E_x \hat{a}_x + E_y \hat{a}_y) \times \omega_c \hat{a}_z].$$

(The minus sign is used for the ions.) We can now determine the $x, y,$ and $z$ components of $\mathbf{V}_1$ to be

$$v_{1x} = a_1 E_x \pm a_2 \omega_c E_y.$$  \hfill (2-100)

$$v_{1y} = \mp a_2 \omega_c E_x + a_1 E_y.$$  \hfill (2-101)

$$v_{1z} = a_0 E_z.$$  \hfill (2-102)
This is expressible in matrix notation as

\[
\begin{bmatrix}
v_{1x} \\ v_{1y} \\ v_{1z}
\end{bmatrix} = \begin{bmatrix}
a_1 & \pm a_2 \omega_c & 0 \\ \mp a_2 \omega_c & a_1 & 0 \\ 0 & 0 & a_0
\end{bmatrix} \times \begin{bmatrix}
E_x \\ E_y \\ E_z
\end{bmatrix}. \tag{2-103}
\]

The lower signs are for the ions. The off-diagonal terms of the matrix are called Hall effect terms from the similar behavior observed in solid state work. Note that this can result in, for example, a velocity in the \(x\) direction due to an electric field in the \(y\) direction. It should also be noted that for ions, the off-diagonal terms change sign. Each subsequent mobility tensor, therefore, may be simply changed to show the ion behavior by changing the signs of the off-diagonal terms and \(q\). Remember that in an isotropic medium, \(V\) is always directly proportional to \(E\) and the proportionality constant is \(\mu\), the mobility, which is a scalar in this case. Equation (2-103) shows a similar relation for an anisotropic material (the plasma in a magnetic field) and the tensor shown is called the mobility tensor.

If case 2 is assumed to be the most general, we can then obtain the mobility tensor by substitution of \(a_0\), \(a_1\), and \(a_2\) into (2-103) as shown in (2-104).

\[
\mu = \begin{bmatrix}
-j \omega q & \frac{1}{m(\omega_c^2 - \omega^2)} & \pm \frac{\omega_c q}{m(\omega_c^2 - \omega^2)} & 0 \\
0 & -j \omega q & \frac{1}{m(\omega_c^2 - \omega^2)} & 0 \\
0 & 0 & 0 & -q \\
\end{bmatrix}. \tag{2-104}
\]

We can factor out \(-q \frac{1}{j \omega m}\) from all terms of equation (2-104) and arrive at (2-105) for the mobility tensor.
\[ \mu = \frac{-q}{j \omega m} \begin{bmatrix} -\omega^2 & \mp j \omega_c \omega & 0 \\ \frac{\omega_f}{\omega_c - \omega^2} & \frac{-\omega^2}{\omega_c^2 - \omega^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \] (2-105)

Up to this point, we have only followed the equation of motion for a single electron or ion. Since plasma is composed of many particles, we would like to obtain the collective effects of electric and magnetic fields on all the particles. We will obtain these effects by considering the electric current produced by all the particles. A simple straightforward approximation to the collective electric current behavior is to assume that the current of \( N \) particles is \( N \) times the electric current of a single one. Thus, the single-particle electric current would be \( I_1 = qv \). The electric current of \( N \) particles would then be

\[ I_N = Nqv, \] (2-106)

The collective electric current density vector is then

\[ J_E = qnV, \] (2-107)

Where \( n \) is the particle density. So, in terms of the mobility,

\[ J_E = qn\mu \cdot E = \sigma \cdot E, \] (2-108)

We have now defined a conductivity tensor such that

\[ \sigma = qn\mu, \] (2-109)

We may convert (2-105) into a conductivity tensor as

\[ \sigma = \frac{j nq^2}{\omega m} \begin{bmatrix} -\omega^2 & \mp j \omega_c \omega & 0 \\ \frac{\omega_f}{\omega_c - \omega^2} & \frac{-\omega^2}{\omega_c^2 - \omega^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \] (2-110)

The conductivity of the plasma when the magnetic field is removed is a scalar, and is the constant that multiplies (2-110). Note also that the conductivity along the magnetic field...
lines is the same as though the field were not present. At this juncture, we note that if the
rule to obtain the ion effect that is changing the sign of \( \omega_c \) and \( q \) is followed, no net
changes are noted in the signs of any diagonal terms in (2-110). This makes sense, since
the ions and electrons making up the plasma act as two conducting fluids that are in
parallel, so their respective conductivities would add directly. The off-diagonal terms do
change sign, which implies the Hall effects of ions and electrons tend to cancel.

We now apply a collisional process to this problem. Since only a collision-less single-
particle model was first assumed, and then a generalization to the mobility of a collection
of particles was made in (2-106) by merely multiplying the single-particle mobility by the
density, we ought to at least consider the interaction between particles in terms of a
momentum transfer collision frequency, to see if we might be able to improve the model.

We can now examine the momentum conservation equation with this effect. It
becomes, for \( V_1 \),

\[
m \frac{dv_1}{dt} + mv_1 v_m = q(E + v_1 \times B).
\]  

(2-111)

The inclusion of the second term on the left-hand side of (2-111) is permitted if we
assume that the particle loses all of its ordered momentum after each collision, that is, \( v_m \)
times per second.

If an \( e^{-j\omega t} \) dependence is assumed, then the left-hand side of (2-111) becomes

\[
-j\omega mv_1 \left( 1 - \frac{v_m}{j\omega} \right) = -j\omega mv_1 \left( \frac{\omega + jv_m}{\omega} \right).
\]  

(2-112)

The result tells us that if we replace \( \omega \) by \( (\omega + jv_m) \) in equation (2-110), we should
obtain the conductivity tensor including the effects of collisions in the form shown in
equation (2-113)
\[ \sigma = \frac{j n q^2}{m(\omega + jv_m)} \begin{bmatrix} \frac{(\omega + jv_m)^2}{(\omega + jv_m)^2 - \omega_c^2} & \frac{\pm j \omega_c (\omega + jv_m)}{(\omega + jv_m)^2 - \omega_c^2} & 0 \\ \frac{j \omega_c (\omega + jv_m)}{(\omega + jv_m)^2 - \omega_c^2} & \frac{(\omega + jv_m)^2}{(\omega + jv_m)^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2-113) \]

Note that since \( m \) appears in the expression for \( \omega_c \), \( \omega_c = qB/m \), the substitution must be made there as well. If \( \omega = 0 \), we can obtain a nonsingular value for the conductivity tensor, since the velocity in the \( z \) direction will not now approach infinity as time gets large. When \( \omega = 0 \), \( \sigma \) is then

\[ \sigma = \frac{n q^2}{mv_m} \begin{bmatrix} \frac{v_m^2}{v_m^2 + \omega_c^2} & \frac{\pm \omega_c v_m}{v_m^2 + \omega_c^2} & 0 \\ \frac{\pm \omega_c v_m}{v_m^2 + \omega_c^2} & \frac{v_m^2}{v_m^2 + \omega_c^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2-114) \]

Note here the off-diagonal terms will again change signs depending upon the sign of the charge. The result with both \( \omega_c = 0 \) and \( \omega = 0 \) is then

\[ \sigma = \frac{n q^2}{mv_m}, \quad (2-115) \]

where \( \sigma \) is a scalar quantity. The conductivity representation is often used in Maxwell’s equations, where it appears as

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \sigma \cdot \mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (2-116) \]

The only effect of the electrons and ions is found in the conductivity term of (2-116). The permittivity is still that of free space. However, a form that is completely equivalent, in which (2-116) is written with zero conductivity and a permittivity different from that of free space, is often used. With the \( e^{-j\omega t} \) notation this is a relatively straightforward situation. Equation (2-116) becomes

\[ \nabla \times \mathbf{H} = \sigma \cdot \mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -j \omega \varepsilon_0 \left( I - \frac{\sigma}{j \omega \varepsilon_0} \right) \mathbf{E}. \quad (2-117) \]
where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the unit tensor and $\varepsilon_0 = \begin{bmatrix} \varepsilon_0 & 0 & 0 \\ 0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 \end{bmatrix}$.

The quantity in the parentheses in (2-117) is the relative permittivity of the plasma and (2-116) may now be modified to present this dielectric constant. To do this, we first compute $\sigma / j\omega \varepsilon_0$ to be

$$\frac{\sigma}{j\omega \varepsilon_0} = \frac{nq^2}{m\omega \varepsilon_0 (\omega + j\nu_m)} \begin{bmatrix} (\omega + j\nu_m)^2 & j\omega c (\omega + j\nu_m) & 0 \\ (\omega + j\nu_m)^2 - \omega_c^2 & (\omega + j\nu_m)^2 - \omega_c^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  (2-118)

The result for $\varepsilon / \varepsilon_0$, which is the permittivity of cold plasma, using (2-116) in (2-117) is then given by Equation (2-119)

$$\frac{\varepsilon}{\varepsilon_0} = \begin{bmatrix} 1 - \frac{nq^2 (\omega + j\nu_m)}{m\omega \varepsilon_0 (\omega + j\nu_m)^2 - \omega_c^2} & \frac{\mp j\omega c nq^2}{m\omega \varepsilon_0 (\omega + j\nu_m)^2 - \omega_c^2} & 0 \\ \frac{\pm j\omega c nq^2}{m\omega \varepsilon_0 (\omega + j\nu_m)^2 - \omega_c^2} & 1 - \frac{nq^2 (\omega + j\nu_m)}{m\omega \varepsilon_0 (\omega + j\nu_m)^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{nq^2}{m\omega \varepsilon_0 (\omega + j\nu_m)} \end{bmatrix}.$$  (2-119)

We may simplify (2-119) by introducing a substitution. Let $\omega_p^2 = \frac{nq^2}{m\varepsilon_0}$ then (2-119) becomes

$$\frac{\varepsilon}{\varepsilon_0} = \kappa = \begin{bmatrix} 1 - \frac{(\omega + j\nu_m) \omega_p^2}{\omega (\omega + j\nu_m)^2 - \omega_c^2} & \frac{\mp j\omega c \omega_p^2}{\omega (\omega + j\nu_m)^2 - \omega_c^2} & 0 \\ \frac{\pm j\omega c \omega_p^2}{\omega (\omega + j\nu_m)^2 - \omega_c^2} & 1 - \frac{(\omega + j\nu_m) \omega_p^2}{\omega (\omega + j\nu_m)^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega (\omega + j\nu_m)} \end{bmatrix}.$$  (2-120)

We call $\omega_p$ the plasma frequency for either electrons or ions, depending upon which species is considered. This frequency is a very fundamental characteristic of all plasmas, and note that it is independent of the sign of the charge.
Similar calculations for ion motion result in the identical form for the ion dielectric constant. The effect of ions may be combined directly with those of the electrons by assuming conductivities in parallel to give a net plasma conductivity and hence a net permittivity. This last expression (2-120) is valid for all three cases of \( \omega \), since the singularities for \( \omega = 0 \) or \( \omega = \omega_c \) are damped out by the collisional effects.

The single-particle model developed here is valid under many conditions (various pressures, temperatures and electron densities) when the cooperative nature of plasma interactions is not applicable. Further extension of validity of plasma behavior now requires a statistical approach.

2.5.1 A Drude-Lorentz Dispersive Model

Equation (2-120) in the absence of an external magnetic field would give us the permittivity of isotropic plasma. In this case the off-diagonal elements of the permittivity tensor become zero and we can use the one-dimensional formula for the permittivity. This equation is also known as Drude-Lorentz model. The one-dimensional equation of motion for a harmonically bound classical electron interacting with an electric field \( \mathbf{E} \) is given by (2-121) [21].

\[
    m(\ddot{\mathbf{r}} + \mathbf{v} \dot{\mathbf{r}} + \omega_0^2 \mathbf{r}) = -e \mathbf{E}(r, t), \tag{2-121}
\]

where \( \omega_0 \) is the natural frequency of the oscillator, and \( v \) is the collision frequency (damping constant). For an incident electromagnetic field of frequency \( \omega \), \( \mathbf{E}(r, t) \) at the point \( \mathbf{r} \) can be conveniently represented by a complex exponential \( \mathbf{E}_0 e^{j\omega t} \). The steady-state solution, in complex form, is given by equation (2-122)

\[
    \mathbf{r} = -\frac{e}{m_e} \left( \omega_0^2 - \omega^2 + j\nu \omega \right)^{-1} \mathbf{E}_0, \tag{2-122}
\]
where the electric dipole moment of the electron, \( \mathbf{p} = -e \mathbf{r} \), corresponds to the macroscopic relation for the polarizability \( \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \). Where \( \chi_e \) is the complex electric susceptibility

\[
\chi_e(\omega) = \frac{Ne^2}{m_e} \left( \omega_0^2 - \omega^2 + j\upsilon\omega \right)^{-1},
\]

\( N \) is the number of polarizable electrons per unit volume. The real part of the susceptibility given in equation (2-124) gives the frequency dependence (dispersion) of the permittivity \( \varepsilon(\omega) = \varepsilon_0 [1 + \chi_1(\omega)] \) and index of refraction \( n(\omega) = \sqrt{1 + \chi_1(\omega)} \)

\[
\chi_1(\omega) = Re[\chi_e(\omega)] \cong -\frac{Ne^2}{2m\omega_0} \frac{(\omega - \omega_0)}{\left(\omega - \omega_0\right)^2 + \frac{\upsilon^2}{4}},
\]

The imaginary part represents the absorption coefficient. This function has the form of a Lorentzian

\[
\chi_2(\omega) = Im[\chi_e(\omega)] \cong -\frac{Ne^2\upsilon}{4m\omega_0} \frac{1}{\left(\omega - \omega_0\right)^2 + \frac{\upsilon^2}{4}}.
\]

In the more accurate quantum theory of dispersion, the frequency \( \omega_0 \) is replaced by a sum over several atomic transition frequencies and the damping parameters \( \upsilon \) are determined by excited-state lifetimes. The real and imaginary parts of the susceptibility are connected by the Kramers–Kronig relations: \( \chi_1(\omega) = \frac{1}{\pi} \varphi \int_{-\infty}^{+\infty} \frac{\chi_2(\omega)}{\omega' - \omega} \, d\omega' \) and \( \chi_2(\omega) = -\frac{1}{\pi} \varphi \int_{-\infty}^{+\infty} \frac{\chi_1(\omega)}{\omega' - \omega} \, d\omega' \), where \( \varphi \) signifies the Cauchy principal value of the integral.

### 2.5.2 Dielectric Constant of Collisional Plasma

This leads us, quite naturally, to ask what the essential distinction is between the response of free electrons in plasma to an electromagnetic wave, and that of free electrons in an ohmic conductor. It turns out that the main distinction is the relative
strength of electron-ion collisions. In the presence of electron-ion collisions, we can model the equation of motion of an individual electron in plasma or a conductor as [21]

\[ m(\ddot{\mathbf{r}} + \mathbf{u} \dot{\mathbf{r}}) = -e \mathbf{E}(r, t) . \]  

(2-126)

Where \( \mathbf{E}(r, t) \) is the wave electric field. The collision term \( i.e., \) the second term on the left-hand side takes the form of a drag force proportional to \( -\mathbf{V} \). In the absence of the wave electric field, this force damps out any electron motion on the typical time-scale \( \nu^{-1} \). Since, in reality, an electron loses virtually all of its directed momentum during a collision with a much more massive ion, we can regard \( \nu \) as the effective electron-ion collision frequency. Assuming the usual \( \exp(-j\omega t) \) time-dependence of perturbed quantities and \( \dot{\mathbf{r}} = \mathbf{V} \), we can solve (2-126) to give

\[ \mathbf{V} = j\omega \mathbf{r} = \frac{-j\omega e \mathbf{E}}{m_e \omega (\omega + j\nu)} , \]  

(2-127)

Hence, the perturbed current density can be written as

\[ \mathbf{J} = -en_e \mathbf{V} = \frac{jn_e e^2 \mathbf{E}}{m_e (\omega + j\nu)} , \]  

(2-128)

Where \( n_e \) is the number density of free electrons. It follows that the effective conductivity of the medium takes the following form.

\[ \sigma = \frac{\mathbf{J}}{\mathbf{E}} = \frac{jn_e e^2}{m_e (\omega + j\nu)} . \]  

(2-129)

Now, the mean rate of ohmic heating per unit volume in the medium is written:

\[ \langle P \rangle = \frac{1}{2} Re(\sigma) E_0^2 \]  

(2-130)

where \( E_0 \) is the amplitude of the wave electric field. Note that only the \textit{real part} of \( \sigma \) contributes to ohmic heating, because the perturbed current must be \textit{in phase} with the wave electric field in order for there to be a net heating effect. An imaginary \( \sigma \) gives a perturbed current which is in phase quadrature with the wave electric field. In this case,
there is zero net transfer of power between the wave and the plasma over a wave period.

We can see from equation (2-129) that in the limit in which the wave frequency is much larger than the collision frequency \( (\omega \gg \nu) \), the effective conductivity of the medium becomes purely imaginary [21]

\[
\sigma \approx \frac{j n_e e^2}{m_e \omega} . \tag{2-131}
\]

In this limit, there is no loss of wave energy due to ohmic heating, and the medium acts like conventional plasma. In the opposite limit, in which the wave frequency is much less than the collision frequency \( (\omega \ll \nu) \), the effective conductivity becomes purely real [21]

\[
\sigma \approx \frac{n_e e^2}{m_e \nu} . \tag{2-132}
\]

In this limit, ohmic heating losses are significant, and the medium acts like a conventional ohmic conductor. By having \( \mathbf{p} = -e \mathbf{r} \) and \( \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \) we can again reach the following equation for collisional plasma permittivity

\[
\varepsilon_r = 1 - \frac{\omega_{pe}^2}{\omega(\omega + j \nu)} = 1 - \frac{\omega_{pe}^2}{\omega^2 + \nu^2} + \frac{\nu}{\omega} \frac{\omega_{pe}^2}{\omega^2 + \nu^2} j = \varepsilon'_r + j \varepsilon''_r . \tag{2-133}
\]

We can derive the following dispersion relation from (2-128)

\[
k^2 c^2 = \omega^2 \varepsilon_r = \omega^2 - \frac{\omega_{pe}^2 \omega}{\omega + j \nu} . \tag{2-134}
\]

It can be seen that, in the limit \( (\omega \gg \nu) \), the above dispersion relation reduces to the dispersion relation for a conventional (collision-less) plasma \( (\omega^2 - \omega_{pe}^2) \). In the opposite limit by using (1-132), we obtain

\[
k^2 = \frac{\omega^2}{c^2} + j \frac{\omega_{pe}^2 \omega}{\omega c^2} = \varepsilon_0 \mu_0 \omega \left( \omega + j \sigma \right) . \tag{2-135}
\]

Of course, the above dispersion relation is identical to the dispersion relation (with \( \varepsilon = 1 \)) for an ohmic conductor. Our main conclusion is that the dispersion relation can be
used to describe electromagnetic wave propagation through both collisional plasma and an ohmic conductor. We can also deduce that in the low frequency limit, \((\omega \ll \upsilon)\), a collisional plasma acts very much like an ohmic conductor, whereas in the high frequency limit, \((\omega \gg \upsilon)\), an ohmic conductor acts very much like a collisionless plasma.

2.6 Perturbation Theory of Resonant Cavities

2.6.1 Introduction

When a small piece of ferrite or dielectric material is introduced into a resonant cavity the frequency of resonance is changed by a small amount and the selectivity of the cavity is lowered. These effects are commonly used in the measurement of the properties of the sample; the relation between the changes in frequency and selectivity and the properties can be derived by perturbation theory and the derivation has been given by several authors. The earliest treatments were given by Bethe and Schwinger, Kahan, and Slater [22], [23], the perturbation considered being a small deformation of the boundary surface of the cavity. Casimir has applied the theory to this case and also to the case of a small body introduced into the cavity [24]. The latter case is applicable to the measurement of dielectric constants of small samples of material at microwave frequencies. Waldron has modified Casimir’s treatment to allow its application to anisotropic magnetic samples, namely ferrites [25], [26]. In all these treatments the aim has been to obtain a formula relating the frequency shift to the magnitude of the deformation, for example, to the dielectric constant of a test sample. It is implicit in [22-24] and explicit in reference [25] that, by taking the frequency shift to be complex, the imaginary part as well as the real part of the property of the sample under test can be measured.
The relation between frequency shift and deformation obtained in the above treatment is now well known. It is, in fact, more accurate than is sometimes thought, but to make full use of this accuracy in measurements on a specimen it is necessary to choose the specimen shape with some care [27]. By using some amended methods, the perturbation theory can be extended for arbitrarily shaped objects and the error can be reduced to a large extent [28-30]. The purpose of this section is to give the derivation of the perturbation formula in detail so as to demonstrate the approximations made, and the absence of others that are sometimes thought to have been made. The circumstances under which the formula may be applied will then be discussed.

2.6.2 Derivation of the Perturbation Formula

In the unperturbed state, in the empty cavity, oscillating in only one of its normal modes lets the electric and magnetic fields in the cavity be

\[
\begin{align*}
E &= E_0 e^{j \omega t}, \\
H &= H_0 e^{j \omega t},
\end{align*}
\]  

(2-136)

where \(E_0\) and \(H_0\) are functions of position. The field configurations are independent of the field magnitudes, always supposing that in normal working these will not be so large as to cause dielectric breakdown in the cavity or appreciable heating of the walls by induced currents. On introducing a small ferrite or dielectric sample into the cavity, the fields and resonance frequency are modified so that we now have

\[
\begin{align*}
E' &= (E_0 + E_1) e^{j (\omega + \delta \omega) t}, \\
H' &= (H_0 + H_1) e^{j (\omega + \delta \omega) t},
\end{align*}
\]  

(2-137)

Where it is supposed that the fields in the perturbed case can be represented as the sums of the unperturbed fields, \(E_0, H_0\), and additional fields, \(E_1, H_1\), with a frequency change...
It will later be necessary to stipulate that \( \delta \omega \ll \omega \) and that over most of the cavity volume \( E_1 \) and \( H_1 \) are small compared with \( E_0 \) and \( H_0 \). It is convenient to make these assumptions at the present stage. It should be noted that it is not required that \( E_1 \) and \( H_1 \) should be small compared with \( E_0 \) and \( H_0 \) in the neighborhood of the sample. (In this section, the neighborhood of the sample will be taken to include the region occupied by the sample itself.)

\( E_0 \) and \( H_0 \) in (2-137) will, in practice, not necessarily be identical with \( E_0 \) and \( H_0 \) in (2-136). The significance of \( E_0 \) and \( H_0 \) in (2-137) is that they have the same configuration as in (2-136) and that they comprise all fields that are expressible as the empty-cavity mode in question. The quantity that is to be measured is the complex frequency shift; this is independent of the amplitudes of oscillation in the perturbed and unperturbed cases, so that, for theoretical purposes, \( E_0 \) and \( H_0 \) of (2-136) may be regarded as multiplied by some scale factor to make them equal to \( E_0 \) and \( H_0 \) of (2-137). The requirement that \( E_1 \) and \( H_1 \) are small compared with \( E_0 \) and \( H_0 \) may then be expressed as the requirement that the field configurations are only slightly distorted, except in the neighborhood of the sample. This is equivalent to requiring \( \delta \omega \) to be small compared with \( \omega \). It is not necessary to stipulate that the energy stored in the sample is small compared with that in the empty cavity, as is sometimes thought. It is sufficient to stipulate that it is a small fraction of the total energy in the cavity in the perturbed condition. This point was first made by Spencer, LeCraw, and Ault [31]. Substituting from equations (2-136) and (2-137), in turn, into the Maxwell’s equation

\[
\nabla \times E = -\frac{\partial B}{\partial t} ,
\]

we obtain

\[
\nabla \times E = -\frac{\partial B}{\partial t} , \quad \text{(2-138)}
\]
∇ × \mathbf{E}_0 = -\frac{∂ \mathbf{B}_0}{∂t} = -j \omega \mathbf{B}_0,

and

∇ × (\mathbf{E}_0 + \mathbf{E}_1) = -j(\omega + \delta \omega)(\mathbf{B}_0 + \mathbf{B}_1),

subtracting,

∇ × \mathbf{E}_1 = -j[\omega \mathbf{B}_1 + \delta \omega(\mathbf{B}_0 + \mathbf{B}_1)]. \quad (2-139)

Similarly, using Maxwell’s equation

∇ × \mathbf{H}_1 = \frac{∂ \mathbf{D}}{∂t}, \quad (2-140)

we find that

∇ × \mathbf{H}_1 = j[\omega \mathbf{D}_1 + \delta \omega(\mathbf{D}_0 + \mathbf{D}_1)]. \quad (2-141)

Forming the scalar product of \mathbf{H}_0 with (2-139) and of \mathbf{E}_0 with (2-140) and adding, we obtain

\mathbf{E}_0 \cdot ∇ × \mathbf{H}_1 + \mathbf{H}_0 \cdot ∇ × \mathbf{E}_1 = j ω [\mathbf{E}_0 \cdot \mathbf{D}_1 - \mathbf{H}_0 \cdot \mathbf{B}_1]

\hspace{1cm} + j δ ω [(\mathbf{E}_0 \cdot \mathbf{D}_0 - \mathbf{H}_0 \cdot \mathbf{B}_0) + (\mathbf{E}_0 \cdot \mathbf{D}_1 - \mathbf{H}_0 \cdot \mathbf{B}_1)]. \quad (2-142)

Here,

\begin{align*}
\mathbf{B}_0 &= \mu_0 \mathbf{H}_0 \quad \mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0 \\
\mathbf{B}_1 &= \mu_0 \mathbf{H}_1 \quad \mathbf{D}_1 = \varepsilon_0 \mathbf{E}_1 \\
\mathbf{D}' &= \mathbf{D}_0 + \mathbf{D}_1 \quad \text{,} \quad (2-143)
\end{align*}

Outside the sample and

\mathbf{D}_1 = \varepsilon_0 [\varepsilon (\mathbf{E}_0 + \mathbf{E}_1) - \mathbf{E}_0], \quad (2-144)

inside the sample. If the sample is of a magnetically isotropic material then

\mathbf{B}_1 = \mu_0 [\mu(\mathbf{H}_0 + \mathbf{H}_1) - \mathbf{H}_0], \quad (2-145)

While if it is a gyromagnetic material,

\mathbf{B}_1 = \mu_0 [\mu(\mathbf{H}_0 + \mathbf{H}_1) - \mathbf{H}_0], \quad (2-146)
inside the sample. Where $\varepsilon$ is the relative permittivity or dielectric constant of the sample; $\mu$ is the relative permeability of an isotropic magnetic sample, while $[\mu]$ is the well-known tensor relative permeability of a ferrite and $\varepsilon_0$ and $\mu_0$ are the permittivity and permeability of free space.

$$\nabla \cdot [(H_0 \times E_1) + (E_0 \times H_1)] \equiv E_1 \cdot \nabla \times H_0 - H_0 \cdot \nabla \times E_1 + H_1 \cdot \nabla \times E_0 - E_0 \cdot \nabla \times H_1$$

which, by virtue of Equations (2-138) and (2-140), may be written

$$H_0 \cdot \nabla \times E_1 + E_0 \cdot \nabla \times H_1 = j \omega (E_1 \cdot D_0 - H_1 \cdot B_0) - \nabla \cdot [(H_0 \times E_1) + (E_0 \times H_1)].$$

Substituting this into the left-hand side of (2-142), we obtain

$$j \omega (E_1 \cdot D_0 - H_1 \cdot B_0) - \nabla \cdot [(H_0 \times E_1) + (E_0 \times H_1)] =$$

$$[j \omega (E_0 \cdot D_1 - H_0 \cdot B_1) + j \delta \omega [(E_0 \cdot D_0 - H_0 \cdot B_0) + (E_0 \cdot D_1 - H_0 \cdot B_1)]]. \quad (2-147)$$

Let $V_0$ be the volume of the cavity and $V_1$ the volume of the sample. Thus $V_0 - V_1$ is the part of the cavity which is not occupied by the sample. Integrate (2-147) over the volume $V_0$:

$$j \omega \iiint_{V_0} (E_1 \cdot D_0 - H_1 \cdot B_0) dV - \iiint_{V_0} \nabla \cdot [(H_0 \times E_1) + (E_0 \times H_1)] dV =$$

$$j \omega \iiint_{V_0} [(E_0 \cdot D_1 - H_0 \cdot B_1)] dV$$

$$+ j \delta \omega \iiint_{V_0} [(E_0 \cdot D_0 - H_0 \cdot B_0) + (E_0 \cdot D_1 - H_0 \cdot B_1)] dV. \quad (2-148)$$

In the region $V_0 - V_1$, (2-143) applies, and the contribution from this region to the integral on the left-hand side of (2-148) is identical with that to the first integral on the right-hand side. Thus we only require the contribution to these integrals from the region $V_1$. For the divergence integral we have, by Green’s theorem,

$$\iiint_{V_0} \nabla \cdot (H_0 \times E_1 + E_0 \times H_1) dV = \iint_{S_0} (H_0 \times E_1 + E_0 \times H_1) \cdot \mathbf{n} dS$$
Where, $S_0$ is the surface of the cavity and $\mathbf{n}$ is a unit vector normal to the element $dS$ of $S_0$. To the extent to which the cavity walls may be regarded as perfectly conducting, $\mathbf{H}_0 \times \mathbf{E}_1$ and $\mathbf{E}_0 \times \mathbf{H}_1$ are tangential to the walls and their scalar product with $\mathbf{n}$ is zero. Thus the divergence integral vanishes.

So far, to the extent to which the cavity may be regarded as having perfectly conducting walls (infinite selectivity), the treatment is exact. The assumption that $\delta \omega$ is much smaller than $\omega$ has been pointed out, but no step has been taken which depends on this assumption. We now make our first approximation, which depends on the smallness of $\delta \omega$, namely to neglect $\mathbf{D}_1$ and $\mathbf{B}_1$ in comparison with $\mathbf{D}_0$ and $\mathbf{B}_0$ in the second integral on the right-hand side of (2-148). This is justified, except in the neighborhood of the sample, by the fact that, when $\delta \omega \ll \omega$, $\mathbf{E}_1$ and $\mathbf{H}_1$ are small compared with $\mathbf{E}_0$ and $\mathbf{D}_0$, bearing in mind also (2-143). In the neighborhood of the sample, the contribution to the integral will be small, provided that the perturbed fields do not depart by more than, say, an order of magnitude from their unperturbed values. This approximation becomes more accurate the smaller $V_1$, and as $V_1$ is made smaller the amount by which the perturbed fields may depart from their unperturbed values, without reducing accuracy, increases. Equation (2-148) now becomes

$$j \omega \iiint_{V_1} (\mathbf{E}_1 \cdot \mathbf{D}_0 - \mathbf{H}_1 \cdot \mathbf{B}_0) dV =$$

$$j \omega \iiint_{V_1} (\mathbf{E}_0 \cdot \mathbf{D}_1 - \mathbf{H}_0 \cdot \mathbf{B}_1) dV + j \delta \omega \iiint_{V_0} (\mathbf{E}_0 \cdot \mathbf{D}_0 - \mathbf{H}_0 \cdot \mathbf{B}_0) dV$$

which can be rearranged in the form of equation (2-149)

$$\frac{\delta \omega}{\omega} = \frac{\iiint_{V_1} [(\mathbf{E}_1 \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}_1) - (\mathbf{H}_1 \cdot \mathbf{B}_0 - \mathbf{H}_0 \cdot \mathbf{B}_1)] dV}{\iiint_{V_0} (\mathbf{E}_0 \cdot \mathbf{D}_0 - \mathbf{H}_0 \cdot \mathbf{B}_0) dV},$$

(2-149)
which is the required formula for the frequency shift. An analogous formula is frequently quoted with a plus sign in the denominator. This is because, with the definitions of $E_0$ and $H_0$ given by (2-136), $E_0$ and $D_0$ both contain a phase factor $j$ if the cavity is oscillating in an $H$ mode (TE), or $H_0$ and $B_0$ both contain a factor $j$ in the case of an $E$ mode (TM). Thus $E_0 \cdot D_0$ and $-H_0 \cdot B_0$ both have the same sign. Where a plus sign appears in the denominator, the factors $j$ have been rendered explicitly.

It is clear from the derivation given in this section that the only condition necessary for (2-149) to hold is that $\delta \omega / \omega$ be small. If the equation is to be of any value, however, it must be possible to evaluate the right-hand side. By virtue of Equations (2-143), (2-144), (2-145), and (2-146), only the fields need be considered; The magnetic flux density and electric displacement field ($B$ and $D$) are readily obtained from the fields if these are known. Note that $E_0$ and $H_0$ present no difficulty, since they are well known for cavities of simple shape.

$E_1$ and $H_1$ can be calculated under certain circumstances. If the sample is in the form of a special case of the general tri-axial ellipsoid and if the microwave field in the cavity at the position of the sample is, in the absence of the sample, sensibly uniform over a volume large compared with that of the sample, the calculation can be made with an accuracy approaching exactness as the volume of the sample approaches zero. This situation is usually obtained in practice; sample shapes commonly used are the sphere and the rod of circular section. The former is an ellipsoid with all three axes equal; the latter appears as an ellipsoid with two equal finite axes and one infinite axis, if the ends of the rod fit flush with the cavity, by virtue of the reflections in the end-faces. The sample is placed at a position of zero electric field for a measurement of permeability, or
of zero magnetic field for a measurement of permittivity. When one field is zero, the other is a maximum and so may be taken as uniform over a limited range.

The calculation of $\mathbf{E}_1$ and $\mathbf{H}_1$ requires that the sample be placed in such a position in the cavity that the modifications it produces in the fields are not influenced by reflection effects in the cavity walls. Attention has been drawn to this point by Spencer, LeCraw, and Reggia [27]. This source of error is avoided if the specimen is kept well away from curved walls, or if curved specimen surfaces are only allowed to come near a plane cavity wall if they intersect orthogonally. Thus a spherical specimen must always be placed well away from cavity walls, while a rod-shaped specimen may be placed with its axis normal to plane walls, the ends fitting flush against the walls. If the rod is shorter than the cavity, it must be much shorter, so that the ends of the rod are not near the walls. A hemispherical specimen could be placed with its plane surface in contact with a plane wall, although this case is not likely to be of practical interest. Sometimes a specimen is used in the form of a disk, but in this case it is not possible to evaluate $\mathbf{E}_1$ and $\mathbf{H}_1$ accurately. The approximation is made that the internal field in the disk is the same as the field, parallel to the plane surfaces of the disk, which exists in the cavity in the absence of the disk. This is true if the disk is a special case of an ellipsoid, for example, if the ratio of thickness to diameter is zero, but this case is of no practical interest. Disks used in practice have values of this ratio of the order of 0.2 or more, and the approximation under these conditions is not very close.

It is not possible to discuss errors quantitatively in a general treatment, although, if necessary, they could be assessed in a particular case. However, provided that the precautions are taken that are described in this section, equation (2-149) will hold to a
greater degree of accuracy than is likely to be attainable practically. In this respect it should be noted that the field in the neighborhood of the specimen being considerably different from the unperturbed field does not detract from the accuracy of (2-149), provided that the sample shape is suitably chosen and the sample is suitably positioned in the cavity. Seidel and Boyet have stated in any particular geometrical and modal situation, the assumption in the perturbation theory is that electric and magnetic fields just outside the sample are their (known) empty cavity values [22], [32].

If this were true, the accuracy of the perturbation formula would not be high, and it would be justifiable to work to a low standard of accuracy. The assumption has not been made, however, in deriving equation (2-149), and if ellipsoidal samples are used, it need not be made. The assumption is made for disk samples, and renders results of measurements on such samples of doubtful accuracy.

A final point bearing on accuracy is the value of the quantity being measured. Usually this will not depart greatly from the value of that property for free space. This may not be the case for a ferrite at ferromagnetic resonance, when the fraction of the cavity energy stored in the specimen may become quite large. Care is needed, therefore, in dealing with this case; however, as long as the sample is sufficiently small, so that $\delta \omega/\omega$ remains small, equation (2-149) is still applicable.


2.7 Perturbation Theory and Its Applications

In the previous section of the discussion on perturbation theory, we have seen how the choice of a model influences the kinds of results that are obtainable when treating a physical problem. By choosing a simple model, a limited amount of information is obtained, while for a more sophisticated model more information can be obtained. For example, if the metal wall of a waveguide is taken to be a perfect conductor, the characteristic equation can be solved for a phase constant $\beta$, which gives no information about the effect of finite conductivity of the metal part of the guide. But if a model is taken which takes the metal to be of finite conductivity, this further information is implicit in the characteristic equation [22].

For any practical waveguide, it is necessary to choose a model that departs in some respect from the physically existing waveguide, and the characteristic equation can therefore not give all the information that is likely to be required. With certain limitations, the extra information can usually be obtained by the use of perturbation theory. This can be used to calculate the effect on the phase constant of a waveguide, or on the frequency of a cavity, of a departure from the mathematical model considered. It is essential that such a departure shall be small, but this must be the case if the model has been well chosen. If the model is not well chosen, the phase constant for the model will not approximate to that of the actual guide, or the model frequency to the actual cavity frequency, and the result is not of much value.

The essence of the perturbation calculation is to determine a small change in the phase constant of a waveguide, or in the resonance frequency of a cavity, due to a small departure of the physical system from the model used. Thus the perturbation method
depends on previously having obtained a solution, for the mode in question, of the characteristic equation. This is one limitation of the method; it is not always possible to obtain, from the characteristic equation for the model. The eigenvalue for a certain mode of the actual system is the phase constant of a guide or the frequency of a cavity. An example of this can be given by the HSP model (Homogeneity, Simplicity and Perfection), for the waveguide which consists of a tube of metal of circular cross-section containing air, does not enable the $E_{p0}$ modes to be treated. The results given by the more general treatment, taking into account the finite conductivity of the metal and the loss tangent of the air, do enable the $E_{p0}$ modes to be investigated.

Another limitation on the use of perturbation theory in the case of waveguides is that it cannot be used near cut-off. As a property of the waveguide varies from its value at cut-off (working frequency, one of the dimensions, dielectric constant of one of the media), the phase constant usually changes rapidly at first, so that quite a small departure, physically, of the physical system from the model can cause a considerable change of the phase constant. Perturbation theory, however, is only applicable when a small change in the physical conditions causes a small change in the eigenvalue.

In section 2.7.1, we shall take as a starting point the perturbation formulae and discuss how they be applied in practice. One of the examples treated exactly in section 2.7.2 will be treated by the perturbation method, and the results compared with those given previously.

### 2.7.1 The Perturbation Formulae

It is convenient to deal first with the perturbation of cavities, and then to develop this
theory to cover the waveguide case. A perturbation consists in changing the material in some part of the cavity. The change may be to replace, in a small volume, the material present in the unperturbed condition by some other material differing considerably in properties. This is the case when a cavity method is used for the measurement of the electric and magnetic properties of small solid samples, and it can be used to treat small changes of shape in the cavity by regarding the change of shape as consisting of the replacement of air by metal or metal by air. Alternatively, the change may consist of replacing, over a large volume (perhaps the whole volume of the cavity), the material present in the unperturbed condition by some other material differing only slightly in properties. This is the case when a cavity method is used to measure the electric and magnetic properties of a gas. Let us suppose that, in the unperturbed condition. The electric and magnetic fields in the cavity are $\mathbf{E}_0$ and $\mathbf{H}_0$ in some mode on which we fix our attention. $\mathbf{E}_0$ and $\mathbf{H}_0$ are supposed known. We also write $\mathbf{D}_0$ and $\mathbf{B}_0$ for the inductions (electric displacement field and magnetic flux density), and let $\omega_0$ be the resonance frequency of the cavity. In the perturbed condition, let the fields become $\mathbf{E}_0 + \mathbf{E}_1$ and $\mathbf{H}_0 + \mathbf{H}_1$, and let the resonance frequency become $\omega_0 + \delta \omega$. $\mathbf{E}_1$ and $\mathbf{H}_1$ are fields that must be added to the unperturbed fields to give the fields in the perturbed cavity. There may be a change in scale factor between the fields in the perturbed and unperturbed conditions; $\mathbf{E}_1$ and $\mathbf{H}_1$ are not intended to take account of this. $\mathbf{E}_0$ and $\mathbf{H}_0$, in the perturbed condition, represent all the energy in the mode in question, while $\mathbf{E}_1$ and $\mathbf{H}_1$ can, in principle, be represented as a sum of all modes except the one obtained in the unperturbed cavity. We also write $\mathbf{D}_0 + \mathbf{D}_1$ and $\mathbf{B}_0 + \mathbf{B}_1$ for the electric displacement field and magnetic flux density in the perturbed cavity.
For the perturbation theory to be valid, it is necessary that over nearly all of the volume of the cavity \( \mathbf{E}_1, \mathbf{H}_1, \mathbf{D}_1, \) and \( \mathbf{B}_1 \) are small compared with \( \mathbf{E}_0, \mathbf{H}_0, \mathbf{D}_0, \) and \( \mathbf{B}_0, \) respectively, although this may not be the case in a very small volume. The perturbation formulae for the cavity may then be stated:

**Case 1: Large Change in Properties over Small Volume**

\[
\frac{\delta \omega}{\omega_0} = \frac{\iiint_{V_1} ((\mathbf{E}_1 \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}_1) - (\mathbf{H}_1 \cdot \mathbf{B}_0 - \mathbf{H}_0 \cdot \mathbf{B}_1)) dV}{\iiint_{V_0} (\mathbf{E}_0 \cdot \mathbf{D}_0 - \mathbf{H}_0 \cdot \mathbf{B}_0) dV},
\]

(2-150)

**Case 2: Small Change in Properties over Large Volume**

\[
\frac{\delta \omega}{\omega_0} = \frac{\iiint_{V_0} ((\mathbf{E}_1 \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}_1) - (\mathbf{H}_1 \cdot \mathbf{B}_0 - \mathbf{H}_0 \cdot \mathbf{B}_1)) dV}{\iiint_{V_0} (\mathbf{E}_0 \cdot \mathbf{D}_0 - \mathbf{H}_0 \cdot \mathbf{B}_0) dV}.
\]

(2-151)

These formulae are first-order approximations which hold to a fairly high degree of accuracy [22] and [29], provided that the conditions under which they are obtained are observed. In Case 1, \( V_1 \) is the volume of small specimen, very much less than \( V_0, \) the volume of the cavity, and \( \mathbf{E}_1, \mathbf{H}_1, \mathbf{D}_1, \) and \( \mathbf{B}_1, \) may depart considerably from zero over the volume \( V_1, \) and outside the sample but still in its neighbourhood, as long as they are nearly zero everywhere else. In case 2, \( \mathbf{E}_1, \mathbf{H}_1, \mathbf{D}_1, \) and \( \mathbf{B}_1, \) must be small everywhere in the cavity. If these conditions are observed, then \( \delta \omega / \omega_0 \) will be small; in practice, only small frequency shifts are observed, and equations (2-150) and (2-151) are then valid.

The evaluation of the denominators of (2-150) and (2-151) presents no difficulty; \( \mathbf{E}_0, \mathbf{D}_0, \mathbf{H}_0, \) and \( \mathbf{B}_0 \) are all known, so that the integration can be carried out. For the numerator, it is necessary to perform a subsidiary calculation to determine \( \mathbf{E}_1, \mathbf{H}_1, \mathbf{D}_1, \) and \( \mathbf{B}_1, \) before the integration can be performed. We shall discuss this question in Section

66
2.7.2 (The Evaluation of $E_1, H_1, D_1, B_1$). Before doing this, however, we now consider the application of the perturbation formulae to waveguides.

For a waveguide, let the volume $V_0$ be the volume of a length of guide equal to one guide wavelength, and let the frequency of waves supplied by a generator be $\omega_0$. This generator is supposed to be at infinity, so that the guide can be considered to be working in a single mode. Now let a perturbation be applied, for example, the guide is filled with a different gas, or a thin rod of dielectric material is placed at its center, or its shape is slightly deformed. At the same time, let the frequency be changed to $\omega_0 + \delta \omega$ so that at this new frequency the guide wavelength is unchanged. The situation in the volume $V_0$ is now analogous to that in the cavity. In fact, the guide can be regarded as an infinitely long cavity. The perturbation is of the cross-section only. The changes made are made over the whole length of the guide. Thus in both numerator and denominator of the perturbation formulae, the integration with respect to $z$ gives the same multiplying factor, which can be cancelled. It is therefore only necessary to integrate over the cross-section, and in place of Eqns. (2-150) and (2-151) we have:

**Case 1: Large Change in Properties over Small Cross-sectional Area**

$$
\frac{\delta \omega}{\omega_0} = \frac{\iint_{S_1} \left( (E_1 \cdot D_0 - E_0 \cdot D_1) - (H_1 \cdot B_0 - H_0 \cdot B_1) \right) dS}{\iint_{S_0} (E_0 \cdot D_0 - H_0 \cdot B_0) dS},
$$

(2-152)

**Case 2: Small Change in Properties over Large Cross-sectional Area**

$$
\frac{\delta \omega}{\omega_0} = \frac{\iint_{S_1} \left( (E_1 \cdot D_0 - E_0 \cdot D_1) - (H_1 \cdot B_0 - H_0 \cdot B_1) \right) dS}{\iint_{S_0} (E_0 \cdot D_0 - H_0 \cdot B_0) dS}.
$$

(2-153)

In the case of the waveguide, however, it is usually required to keep the frequency fixed and to calculate the change in phase constant. We thus consider a further change; in the
perturbed state, the frequency is returned to $\omega_0$, and correspondingly the phase constant changes from $\beta_0$, its value in both the unperturbed state at frequency $\omega_0$ and in the perturbed state at frequency $\omega_0 + \delta \omega$, to $\beta_0 + \delta \beta$. Then

$$\delta \beta = -\left(\frac{d\beta}{d\omega}\right)_{\omega_0} \delta \omega,$$

which can be rearranged as (2-155)

$$\frac{\delta \beta}{\beta_0} = -\left(\frac{d\beta}{d\omega}\right)_{\omega_0} \frac{\delta \omega \omega_0}{\omega_0 \beta_0}.$$

The minus sign arises because the change in frequency is now $-\delta \omega$. It is sometimes convenient to write

$$\left(\frac{d\beta}{d\omega}\right)_{\omega_0} = \frac{d\beta_0}{d\lambda_0} \left(\frac{d\lambda_0}{d\omega}\right)_{\omega_0},$$

in order to evaluate equation (2-155), $\lambda_0$ being the free-space wavelength.

2.7.2 The Evaluation of $E_1, H_1, D_1, B_1$

The evaluation of $E_1, H_1, D_1, B_1$ is not always possible, and then the perturbation method breaks down. Even where it is possible, the calculation may be difficult, but the difficulty is offset by the fact that once the calculation is made it can be applied in a large number of cases.

In measuring the properties of small samples of solid material it is convenient to place the sample at a point in a cavity where either the electric or the magnetic field is zero; for example, at the center of rectangular cavity for mode $TM_{110}$. The magnetic or electric properties, respectively, can then be examined separately. It happens that when one field is zero, the other is the maximum, and is therefore approximately uniform over a limited region. If the sample is small compared with this region, the field inside it can be
calculated as if it were due to a uniform external field. (External means, not the field outside the sample, but the field that existed at the position of the sample before the sample was introduced. The word ‘external’ is justified because this is still the value of the field at large distances. It is the field external to the neighborhood of the sample rather than to the sample itself.) The calculation of the internal field can be performed only if the specimen is in the form of an ellipsoid; special cases of the general ellipsoid that are commonly used are the sphere and the rod of circular cross-section. Solutions have been given for the electric and magnetic fields in dielectric and ferrite specimens of a variety of shapes, in terms of the external field. One other case that is tractable is the rod of any cross-section, if the (uniform) microwave field is in the direction of its axis, when the field in the specimen is equal to the external field. An example of the calculation for a small solid sample is given in the following section.

The case of a small change in properties over a large volume is easier to treat. For example, it may be required to calculate the attenuation coefficient of a waveguide consisting of a metal tube containing air, due to the dielectric loss of the air. The perturbation is the change, over the whole of the waveguide (if the metal is regarded as a perfect conductor), of the dielectric constant from 1 to $1 - j \varepsilon''$. Parameters $E$, $H$, and $B$ are unchanged, so that $E_1$, $H_1$, and $B_1$ are zero, and $D_1$ is easily calculated, since

$$D_0 + D_1 = \varepsilon_0 \varepsilon E_0 = \varepsilon_0 (1 - j \varepsilon'') E_0 ,$$

$$D_1 = \varepsilon_0 (1 - j \varepsilon'') E_0 - \varepsilon_0 E_0 = -j \varepsilon'' \varepsilon_0 E_0 .$$

We shall treat this sample in section 2.7.4 (Dielectric Loss in a Sample Waveguide), and compare the result with that obtained by a more general approach.
It is clear that the possibility of a perturbation approach to a problem depends on the possibility of determining $\mathbf{E}_1$ and $\mathbf{H}_1$; if these can be calculated, so can $\mathbf{D}_1$ and $\mathbf{B}_1$. The kinds of problem that are tractable are therefore the calculation of the frequency shift of a cavity due to the placing of a small ellipsoidal specimen in a region of uniform field; the calculation of the frequency shift of a cavity due to the introduction of a gas which fills the cavity; the calculation of the finite $Q$ of a cavity due to the finite conductivity of the walls; the calculation of the frequency shift of a cavity due to a small deformation of its surface, if the deformation is of a suitable shape; and the calculation of the change of phase constant of a waveguide due to the same kinds of changes. In the waveguide case, the change must be uniform over the whole length of the guide.

We may note here that the well-known method of treating losses in waveguides due to the finite conductivity of the walls is an example of the application of perturbation theory, although it is not usually stated to be such [33], [34].

2.7.3 Frequency Shift of a Cavity due to a Small Solid Specimen

As an example, consider a sphere of dielectric material of relative permittivity $\varepsilon$ placed in a cavity in such a way that the external microwave electric field is uniform and transverse in the neighborhood of the sphere. One way of achieving this situation is to use a rectangular cavity for an $E_{mn0}$ mode, with $m$ and $n$ odd and the sample at the center of the cavity (Figure 2.5). In the $E_{mn}$ modes ($TM_{mnp}$), the field components are
\[
\begin{align*}
E_z &= \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{p\pi z}{c} \\
E_x &= \frac{-1}{K^2} \frac{m\pi v}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{p\pi z}{c} \\
E_y &= \frac{-1}{K^2} \frac{n\pi v}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \frac{p\pi z}{c} \\
H_x &= \frac{j\omega \varepsilon_0}{K^2} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \cos \frac{p\pi z}{c} \\
H_y &= \frac{-j\omega \varepsilon_0}{K^2} \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{p\pi z}{c}
\end{align*}
\]

(2-159)

where \(a, b\) and \(c\) are the dimensions of the cavity in the \(x, y,\) and \(z\) directions respectively. For \(p = 0\), \(E_x\) and \(E_y\) vanish identically. For odd \(m\) and \(n\), \(\cos(m\pi x/a)\) and \(\cos(n\pi y/b)\) vanish at the center of the cavity, where \(x = a/2, y = b/2\). At the centre, therefore, the only field component is \(E_z\), which is a maximum and approximately uniform.

Figure 2.5 Electric field in a rectangular cavity excited in an \(E_{1n0}\) mode both with and without a spherical sample.

If a sphere of dielectric material is immersed in an electric field which in the absence of the sphere is uniform and equal to \(E\), and (Figure 2.6), the field \(E'\) in the sphere is in the same direction as \(E\), and of magnitude

\[
E' = \frac{3E}{\varepsilon + 2}.
\]

(2-160)

This formula holds for microwave fields as long as the sphere is sufficiently small for the field to be sensibly uniform over a region large compared with the sphere. From this, \(E_1\)
and $\mathbf{D}_1$ can be calculated and used in equation (2-162). $\mathbf{E}$ is the value of $\mathbf{E}_0$ at the position of the sphere in the volume $V_1$ while $\mathbf{E}'$ is the value of $\mathbf{E}_0 + \mathbf{E}_1$ in $V_1$. Hence

$$\mathbf{E}_1 = \mathbf{E}' - \mathbf{E}_0 = \mathbf{E}_0 \left\{ \frac{3}{\varepsilon+2} - 1 \right\} = -\mathbf{E}_0 \left\{ \frac{\varepsilon-1}{\varepsilon+2} \right\}. \quad (2-161)$$

For $\mathbf{D}_1$, we have $\mathbf{D}_0 + \mathbf{D}_1 = \varepsilon \varepsilon_0 (\mathbf{E}_0 + \mathbf{E}_1) = \varepsilon \varepsilon_0 \mathbf{E}_0 \frac{3}{\varepsilon+2}$, so that

$$\mathbf{D}_1 = \varepsilon \varepsilon_0 \mathbf{E}_0 \frac{3}{\varepsilon+2} - \varepsilon_0 \mathbf{E}_0 = 2 \varepsilon_0 \mathbf{E}_0 \left\{ \frac{\varepsilon-1}{\varepsilon+2} \right\}. \quad (2-162)$$

In the numerator of (2-150), the second bracket vanishes because $\mathbf{H}_0$, and therefore $\mathbf{B}_0$, are zero at the position of the sample. From (2-161), (2-162) and (2-144), $\mathbf{E}_1$ and $\mathbf{D}_1$ can be replace in (2-150) the first bracket is

$$\mathbf{E}_1 \cdot \mathbf{D}_0 - \mathbf{E}_0 \cdot \mathbf{D}_1 = \varepsilon \mathbf{E}_0^2 \left\{ \frac{-\varepsilon-1}{\varepsilon+2} - 2 \frac{\varepsilon-1}{\varepsilon+2} \right\} = -3 \varepsilon \varepsilon_0 \mathbf{E}_0^2 \frac{\varepsilon-1}{\varepsilon+2},$$

substituting this into (2-150), we obtain

$$\frac{\delta \omega}{\omega_0} = \frac{-3(\varepsilon-1)}{(\varepsilon+2)} \frac{\iiint_{V_1} \mathbf{E}_0^2 \, dV}{\iiint_{V_0} (\mathbf{E}_0 \cdot \mathbf{D}_0 - \mathbf{H}_0 \cdot \mathbf{B}_0) \, dV}. \quad (2-163)$$
On performing the integrations, a constant is obtained which depends on the geometry of the cavity and the mode of oscillation. Thus we have

$$\frac{\delta \omega}{\omega_0} = -A \left\{ \frac{\varepsilon - 1}{\varepsilon + 2} \right\}, \quad (2-164)$$

the negative sign indicates that $\delta \omega$ is negative. On introducing the sphere the frequency is lowered. This is as we should expect, since to increase the dielectric constant of a medium lowers the wavelength, so that relative to the wavelength the space occupied becomes larger. From (2-164) the dielectric constant can be calculated if $\delta \omega / \omega_0$ is measured. Dielectric constant of practical materials are complex, and by regarding the frequency also as complex, (2-164) can be applied to the measurement of both real and imaginary parts of $\varepsilon$. Let $Q_0$ be the $Q$ of the cavity in the unperturbed condition, and let us define its complex resonance frequency, $\Omega_0$, by

$$\Omega_0 = \omega_0 (1 + j/2Q_0), \quad (2-165)$$

In the perturbed condition, let $\Omega$ change from $\Omega_0$ to $\Omega_0 + \delta \Omega$, with real part $\omega_0 + \delta \omega$, and let the $Q$ now be $Q_1$, then

$$\Omega_0 + \delta \Omega = (\omega_0 + \delta \omega)(1 + j/2Q_1). \quad (2-166)$$

Subtracting equation (2-165) from (2-166), and retaining small terms only of the first order, we obtain

$$\delta \Omega = \delta \omega + j\omega_0 \left\{ \frac{1}{2Q_1} - \frac{1}{2Q_0} \right\},$$

that can be rearranged as (2-167)

$$\frac{\delta \Omega}{\omega_0} = \frac{\delta \omega}{\omega_0} + j \left\{ \frac{1}{2Q_1} - \frac{1}{2Q_0} \right\}. \quad (2-167)$$

Writing (2-164) in terms of the complex frequency, we obtain

$$\frac{\delta \Omega}{\omega_0} = -A \left\{ \frac{\varepsilon' - j\varepsilon'' - 1}{\varepsilon' - j\varepsilon'' + 2} \right\} \times \left\{ \frac{(\varepsilon' + 2) + j\varepsilon''}{(\varepsilon' + 2) + j\varepsilon''} \right\} \quad (2-168)$$
which becomes, for small $\varepsilon''$,

$$\frac{\delta \omega}{\omega_0} = -\frac{A}{\varepsilon' + 2}\left\{\varepsilon' - 1 - \frac{3j\varepsilon''}{\varepsilon' + 2}\right\}. \quad (2-169)$$

Substituting for $\delta \Omega / \omega_0$ from (2-167), and equating real and imaginary parts, we finally obtain

$$\frac{\delta \Omega}{\omega_0} = -A \left\{\frac{(\varepsilon' - 1)}{(\varepsilon' + 2)}\right\}, \quad (2-170)$$

$$\frac{1}{Q_1} - \frac{1}{Q_0} = \frac{6A \varepsilon''}{(\varepsilon' + 2)^2}.$$

The first line of (2-170) shows that as long as $\varepsilon''$ is small the value of $\varepsilon$ calculated from (2-164) is in fact the real part of $\varepsilon$. The imaginary part of the permittivity, $\varepsilon''$, can be calculated from the second line of (2-170) if the $Q$’s of the unperturbed and perturbed cavity are measured as well as the shift of resonance frequency. While only a spherical sample has been considered here, analogous calculations can be made for samples of other shapes, both dielectrics and ferrites [35-39].

### 2.7.4 Dielectric Loss in a Simple Waveguide

The simple waveguide to be considered here consists of a metal tube of circular cross-section, containing air. Exact treatments of this guide can be done by using two models. For the first of these, the metal was taken to have the properties of a perfect conductor and the air those of vacuum. For the second, the metal was taken to have large but finite conductivity and the permeability of free space, and to extend to infinity, while the air was taken to have a dielectric constant $1 - j\varepsilon''$, with $\varepsilon'' \ll 1$. Using the second model, it was shown how the attenuation, due to dielectric losses in the air and conduction losses in the metal, could be calculated. We shall now calculate the attenuation due to dielectric
loss in the air, regarding air as a perturbation of the vacuum of the first model, and compare the perturbation result with the first-order result obtained by the use of the second model.

Equation (2-153) gives the appropriate perturbation formula. The second bracket of the numerator of the right-hand side vanishes because there is no magnetic perturbation, so that $H_1$ and $B_1$ are zero. It was shown in an earlier section that $E_1 = 0$, and $D_1 = -j\varepsilon''\varepsilon_0 E_0$. Thus (2-153) becomes

$$\frac{\delta \omega}{\omega_0} = +j\varepsilon''\varepsilon_0 \frac{\iint_{S_0} E_0^2 dS}{\iiint_{S_0} (\varepsilon_0 E_0^2 - \mu_0 H_0^2) dS}. \quad (2-171)$$

Now, the electric energy per unit length of the waveguide is $\frac{\varepsilon_0}{2} \iiint_{S_0} E_0^2 dS$, while the magnetic energy per unit length is $-\frac{\mu_0}{2} \iiint_{S_0} H_0^2 dS$. Also, the electric and magnetic energies are equal. Thus the integral in the numerator of the right-hand side of equation (2-171) is half that in the denominator, and we have

$$\frac{\delta \omega}{\omega_0} = \frac{j\varepsilon''}{2}. \quad (2-172)$$

It will be noticed that $\delta \omega/\omega_0$ depends only on $\varepsilon''$, not on the geometry of the waveguide nor on the mode of operation. At constant frequency, it is $\delta \beta/\beta_0$ that is required, and this is obtained from (2-172) by the use of (2-154). It is well known that, for a simple waveguide of the type we are considering,

$$\frac{\omega}{\beta} = \frac{c}{\sqrt{1-(\omega_{co}/\omega)^2}}, \quad (2-173)$$

where $\omega_{co}$ is the cut-off value of $\omega$. Differentiating,

$$\frac{d\beta}{d\omega} = \frac{1}{c\sqrt{1-(\omega_{co}/\omega)^2}}, \quad (2-174)$$
and substituting from (2-173) and (2-174) into (2-155), at frequency $\omega_0$, we obtain

$$\frac{\delta \beta}{\beta_0} = \frac{-1}{1-(\omega_0/\omega)^2} \frac{\delta \omega}{\omega},$$

and from (2-171) we obtain finally

$$\frac{\delta \beta}{\beta_0} = \frac{-j\epsilon''/2}{1-(\omega_0/\omega)^2}. \quad (2-175)$$

This gives $\delta \beta$ as negative and imaginary, and $\delta \beta$ is the attenuation constant. The result obtained by use of a more sophisticated model [40], [41] was

$$\delta \bar{\beta} = -j \epsilon''/2 \bar{\beta}_0. \quad (2-176)$$

where $\bar{\beta} = \lambda_0 \beta/2\pi$, $\delta \bar{\beta} = \lambda_0 \delta \beta/2\pi$. From (2-173),

$$\bar{\beta}_0 = \sqrt{1 - \omega_0^2/\omega_{co}^2}, \quad (2-177)$$

thus for (2-175) we may write

$$\frac{\delta \bar{\beta}}{\bar{\beta}_0} = \frac{-j\epsilon''/2}{\bar{\beta}_0^2},$$

$$\delta \bar{\beta} = -j \epsilon''/2 \bar{\beta}_0. \quad (2-178)$$

in which $\delta \bar{\beta}$ is the same as (2-176). Thus perturbation theory gives the same result as the first-order expansion of the exact theory, and can be used if $\bar{\beta}_0$ is known. It may be, however, that $\bar{\beta}_0$ cannot be determined for certain modes from the model used and then, of course, (2-178) is of no help. This is the case, for example, with the $E_{\rho 0}$ modes of the circular guide.

Perturbation theory is only valid if $\delta \beta/\beta_0$ is small, so that $\beta_0$ must not be near zero. This applies generally, not only to the example treated above. Thus perturbation theory can only be applied to the waveguides well away from cut-off, or $\omega_0 - \omega_{co}$ must not be very small compared with $\omega_{co}$. 

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2.8 Cavity Design and the Coupling

In Chapter 3 we use the cavity perturbation method. In order to use a cavity we need to couple energy to the cavity. For this research the aperture coupled cavity is used by deploying a rectangular slot however in this section we discuss both aperture coupled and probe coupled cavities. The formulation of the radiation from currents in a waveguide in terms of radiation from equivalent electric and magnetic dipoles is directly applicable to the coupling of waveguide by small apertures, or holes in a common wall. To a first approximation a small aperture in a conducting wall is equivalent to an electric dipole normal to the aperture and having strength proportional to the normal component of the exciting electric field, plus a magnetic dipole in the plane of the aperture and having a strength proportional to the exciting tangential magnetic field. The constants of proportionality are parameters that depend on the aperture size and shape. These constants are called electric and magnetic polarizabilities of the aperture and characterize the coupling or radiating properties of the aperture. A qualitative argument to demonstrate the physical reasonableness of these properties of an aperture is given below.

Figure 2.7a illustrates the normal electric field of strength $\mathbf{E}$ at a conducting surface without an aperture. When an aperture is cut in the screen, the electric field lines fringe through the aperture in the manner indicated in Figure 2.7b. But this field distribution is essentially that produced by an equivalent electric dipole as shown in Figure 2.7c. Note that the dipole is oriented normal to the aperture.

In a similar manner the tangential magnetic field lines shown in Figure 2.7d will fringe through the aperture as in Figure 2.7e. Theses fringing field lines are equivalent to those produced by a magnetic dipole located in the plane of the aperture.
In Bethe’s original theory the dipole moments are determined by the field in the waveguide in the absence of the aperture. Thus, for a circular aperture of radius \( r_0 \ll \lambda_0 \), the dipole moments are related to the incident fields as follows [42]

\[
P = -\varepsilon_0 \alpha_e (\hat{n} \cdot \mathbf{E}) \hat{n},
\]

\[
M = -\alpha_m \mathbf{H}_t,
\]

(2-179a) (2-179b)

where \( \hat{n} \cdot \mathbf{E} \) is the normal electric field and \( \mathbf{H}_t \) is the tangential magnetic field at the center of the aperture. The electric polarizability \( \alpha_e \) is given by

\[
\alpha_e = -\frac{2}{3} r_0^3 \, ,
\]

(2-180a)

and the magnetic polarizability \( \alpha_m \) is given by

\[
\alpha_m = \frac{4}{3} r_0^3 \, .
\]

(2-180b)

The presence of an aperture also perturbs the field on the incident side of the screen. This perturbed field is that radiated by the equivalent dipole which is the negative of those given by (2-179) and located on the input side of the screen. It is important to note that when the aperture is replaced by the equivalent electric dipoles, the field radiated by these is computed by assuming that the aperture is now closed by a conducting wall. The equivalent dipoles correctly account for the field coupled through the aperture in the conducting screen. Bethe’s theory does not lead to an equivalent circuit for the aperture that includes a conductance to represent power coupled through the aperture. The reason for this is that the field assumed to excite the dipoles is chosen as the unperturbed incident field in the waveguide. In actual fact one should use the sum of the incident field and excited field as the polarizing field. Since the excited field is small, the correction to the dipole moments is also small. However, by including the excited dominant modes (the propagating modes) as part of the polarizing field, we will obtain the needed
correction which results in a conductance element in the equivalent circuit. The dominant-mode fields react back on the dipole to account for the radiation of power by the dipoles. Thus, in place of (2-179), the following expressions are used for the dipole strengths.

\[
P = -\varepsilon_0 \alpha_e \left[ \hat{n} \cdot \mathbf{E}_g - \hat{n} \cdot \mathbf{E}_r \right] \hat{n}, \quad (2-181a)
\]

\[
M = -\alpha_m \left[ \mathbf{H}_g + \mathbf{H}_r - \mathbf{H}_2 \right] t. \quad (2-181b)
\]

For the radiation into the input waveguide,

\[
P = \varepsilon_0 \alpha_e \left[ \hat{n} \cdot \mathbf{E}_g - \hat{n} \cdot \mathbf{E}_r \right] \hat{n}, \quad (2-182a)
\]

\[
M = \alpha_m \left[ \mathbf{H}_g + \mathbf{H}_r - \mathbf{H}_2 \right] t. \quad (2-182b)
\]

Figure 2.7 Aperture in a conducting wall. (a) Electric field perpendicular to the wall without an aperture, (b) Electric field lines fringe through the aperture, (c) Replacing the aperture with an equivalent electric dipole, (d) Magnetic field parallel to the wall without an aperture, (e) Magnetic field fringe through the aperture, (f) Replacing the aperture with a magnetic dipole.
Where the generator fields, $E_{g1}$ and $H_{g1}$, are the dominant-mode fields in the input waveguide in the absence of the aperture, $E_{1r}$, $H_{1r}$ are the dominant-mode fields radiated by the dipoles in the input waveguide, and $E_{2r}$, $H_{2r}$ are the dominant-mode fields radiated by the dipoles in the output waveguide. The unit vector $n$ is normal to the aperture and directed from the input waveguide to the output waveguide. The subscript $t$ denotes the tangential component of the magnetic fields. The field resulting from the aperture is determined by closing the aperture by an electric perfectly conducting surface and calculating the fields radiated by the dipoles given above and located at the center of the circular aperture.

The theory is readily extended to include noncircular apertures. However, the procedure outlined above is restricted to circular apertures in a very thin common wall between two waveguides. There is considerable attenuation in the coupling through an aperture in a thick wall and in many practical applications this attenuation must be taken into account.

The examples discussed next will illustrate the application of small-aperture coupling theory to an aperture in a transverse wall and an aperture in the broad wall between two identical rectangular waveguides.

2.8.1 Aperture in a Transverse Wall

Figure 2.8a illustrates a small circular aperture in a transverse wall in a rectangular waveguide. To determine the exciting generator field, assume that the aperture is closed. A $TE_{10}$ mode incident from $z < 0$ is reflected by the conducting wall at $z = 0$ to produce a standing-wave field in the region $z < 0$. This field is as equation (2-183)
\[ E_y = C(e^{-j\beta z} - e^{j\beta z}) \sin \frac{\pi x}{a}, \]  \hspace{1cm} (2-183a)

\[ H_x = -CY_\omega (e^{-j\beta z} + e^{j\beta z}) \sin \frac{\pi x}{a}, \]  \hspace{1cm} (2-183b)

where \( C \) is a constant. There is a \( z \) component of magnetic field which is not required to be known for the present problem. The normal electric field at the aperture is zero; so no induced electric dipole is produced. The tangential magnetic field at the center of the aperture is, from (2-183b)

\[ H_x = -2CY_\omega, \]  \hspace{1cm} (1-184)

![Diagram](image)

Figure 2.8 Aperture in a transverse waveguide wall. (a) Aperture, (b) equivalent magnetic dipole of the aperture and its image, (c) doubling the magnetic dipole and removing the wall.

and hence an induced \( x \)-directed magnetic dipole \( \mathbf{M} \) is produced. In order to determine the total polarizing field using (2-181b) and (2-182b), we must find the fields \( \mathbf{H}_{1r} \) and \( \mathbf{H}_{2r} \) radiated into guide 1 and 2 by a magnetic dipole \( M\mathbf{a}_x \). The field radiated into the
region $z > 0$ is that radiated by the magnetic dipole $\mathbf{M}$, as illustrated in Figure 2.8b. This dipole is equivalent to a half circular current loop in the $yz$ plane as illustrated. To find the field radiated by this dipole in the presence of the conducting transverse wall, image theory may be used. Since the image of the half circular current loop in the transverse wall is the other half of the current loop, the image of $\mathbf{M}$ is another magnetic dipole of moment $\mathbf{M}$. The effect of the transverse wall is equivalent to removing the wall and doubling the strength of the dipole, as depicted in Figure 2.8c. If the field radiated into the region $z > 0$ is

$$
E_y^+ = A e^{-j \beta z} \sin \frac{\pi x}{a} = A e^{-j \beta z} , \tag{2-185a}
$$

$$
H_x^+ = -AY_\omega e^{-j \beta z} \sin \frac{\pi x}{a} = Ah_x e^{-j \beta z} , \tag{2-185b}
$$

then formula for radiation from the current loop is

$$
C_n^+ = \frac{j \omega}{P_{n}} \mathbf{B}_{n}^- \cdot \mathbf{M} , \tag{2-186a}
$$

$$
C_n^- = \frac{j \omega}{P_{n}} \mathbf{B}_{n}^+ \cdot \mathbf{M} . \tag{2-186b}
$$

Replacing (2-185) in (2-186) gives

$$
A = \frac{j \omega \mu_0}{P_{10}} H_x^- (2M) , \tag{2-187}
$$

since the magnitude of field $\mathbf{B}_{n}^-$ is $-\mu_0 H_x = \mu_0 Y_\omega \sin(\pi x/a)$ in the present case. The constant $P_{10}$ is given by

$$
P_{10} = -2 \int_0^a \int_0^b e_y h_x dy dx = 2Y_\omega \int_0^a \int_0^b \sin^2 \frac{\pi x}{a} dy dx = ab Y_\omega , \tag{2-188}
$$

hence we obtain

$$
A = \frac{j \omega \mu_0}{ab} 2M = \frac{j k_0 z_0}{ab} 2M . \tag{2-189}
$$
The presence of the aperture causes a field to be scattered into the region \( Z < 0 \) also. For radiation into this region, the effective magnetic dipole moment is the negative of that used to obtain (2-189). Application of (2-186) now gives

\[
E_y = A \sin \frac{\pi x}{a} e^{j \beta z}, \quad Z < 0, \tag{2-190}
\]

for the radiated field in the input waveguide and where \( A \) is given by (2-189). As expected, the magnetic dipole \( M \text{a}_x \) acts as a shunt source. The \( x \) component of the radiation reaction fields \( H_{1r} \) and \( H_{2r} \), at the center of the aperture, are

\[
H_{1rx} = AY_\omega = \frac{j k_0 Z_0}{abZ_\omega} 2M, \tag{2-191}
\]

\[
H_{2rx} = -AY_\omega = -\frac{j k_0 Z_0}{abZ_\omega} 2M, \tag{2-192}
\]

and the generator field is

\[
H_{g1r} = -2CY_\omega, \tag{2-193}
\]

Since \( M \) represents the dipole strength for radiation into guide 2, we use these fields in (2-181b) to obtain

\[
M = -\alpha_m \left[ -2CY_\omega + \frac{j k_0 Z_0}{abZ_\omega} M \right]. \tag{2-194}
\]

Equation (2-194) can be solved for \( M \) to give

\[
M = \frac{2\alpha_m Y_\omega C}{1 + \frac{j k_0 Z_0}{abZ_\omega} \alpha_m}. \tag{2-195}
\]

We can now complete the evaluation of the constant \( A \) by using this expression for \( M \) in (2-189); thus

\[
A = \frac{4\frac{j k_0 Z_0}{abZ_\omega} \alpha_m}{1 + \frac{4j k_0 Z_0}{abZ_\omega} C}. \tag{2-196}
\]

and the total electric field in the input waveguide is
\[
E_y = [Ce^{-j\beta z} + (A - C)e^{j\beta z}] \sin \frac{\pi x}{a},
\]

so the input reflection coefficient becomes
\[
\Gamma = \frac{A-C}{C},
\]

and the input normalized admittance is
\[
\bar{Y}_{in} = j\bar{B} + \bar{G} = \frac{1-\Gamma}{1+\Gamma}.
\]

When we substitute \( A \) from (2-196) into the expression for \( \Gamma \), we find that
\[
\bar{Y}_{in} = \frac{2-A/C}{A/C} = 1 - j \frac{3ab}{8r^3 \beta}.
\]

The equivalent circuit of the aperture, as seen from the input waveguide, is a normalized shut conductance of unit value plus a shut inductive susceptance. The conductance term is called the radiation conductance and accounts for the power coupled, or radiated, into the output waveguide. The amplitude of the transmitted electric field is \( A \) which is given by (2-196). The transmission coefficient is \( A/C \). From the equivalent circuit the transmission coefficient is \( 1 + \Gamma = 2/(1 + \bar{Y}_{in}) \) which gives the same result. The aperture is equivalent to an inductive susceptance connected across the transmission line. The conductance term represents the output transmission line terminated in a matched load.

### 2.8.2 Aperture-Coupled Cavity

As an example of an aperture-coupled cavity, consider the rectangular cavity coupled to a rectangular guide by means of a small centered circular hole in the transverse wall at \( z = 0 \), as illustrated in Figure 2.9. As indicated earlier, a small circular aperture in a
transverse wall behaves as a shunt inductive susceptance with a normalized value given by (2-198) as

\[
\bar{B}_L = \frac{3ab}{8\beta r_0^3},
\]

where \(a\) is the guide width, \(b\) is the guide height, \(r_0\) is the aperture radius, and \(\beta = [k_0^2 - (\pi/a)^2]^{1/2}\) is the propagation constant for the \(TE_{10}\) mode. The equivalent circuit of the aperture-coupled cavity is thus a short circuited transmission line of length \(d\) shunted by a normalized susceptance \(\bar{B}_L\).

To analyze this coupled cavity, we shall assume initially that there are no losses. A modification is required when small losses are present as given later. The cavity will exhibit an infinite number of resonances, and the input impedance \(\bar{Z}_{in}\) will have an infinite number of zeros interlaced by an infinite number of poles, this being the general behavior of a distributed parameter one-port microwave network. If we are interested in a resonance corresponding to a high value of \(\bar{Z}_{in}\), infinite in the case of no loss, we should examine the nature of \(\bar{Z}_{in}\) in the vicinity of one of its poles. This case corresponds to a parallel resonant circuit.

The input impedance is given by the parallel impedance of \(j\bar{X}_L\) and \(j \tan \beta d\) and is

![Diagram](image_url)
\[
\tilde{Z}_{in} = \frac{-\bar{X}_L \tan \beta d}{j\bar{X}_L + j \tan \beta d},
\]

(2-199)

where \(j\bar{X}_L = (-j\bar{B}_L)^{-1}\). The antiresonances occur when the denominator vanishes, for instance at the poles of \(\tilde{Z}_{in}\), or when

\[
\bar{X}_L = -\tan \beta d = \frac{8\pi^3 \beta d}{3abd}.
\]

(2-200)

To solve this equation for the values of \(\beta\) that yield resonances, graphical methods are convenient. By plotting the two sides of (2-200) as functions of \(\beta d\), the points of intersection yield the solutions for \(\beta d\). When \(\beta\) is known, the resonant frequency may be found from the relation

\[
\frac{\omega}{2\pi} = \frac{c}{2\pi} \left[\beta^2 + \left(\frac{\pi}{a}\right)^2\right]^{1/2}.
\]

(2-201)

This graphical solution is illustrated in Figure 2.10. Note that there are an infinite number of solutions. Normally, \(\bar{X}_L\) is very small, so that the value of \(\beta d\) for the fundamental mode is approximately equal to \(\pi\). The higher-order modes occur for \(\beta d = \beta_n d \approx (n - \frac{1}{2})\pi\), when \(n\), an integer, is large.

Figure 2.10 Graphical solution for resonant frequency of aperture-coupled cavity.
The value of $\beta$ for the first mode will be denoted by $\beta_1$, and the corresponding value of $\omega$ by $\omega_1$, as determined by putting $\beta = \beta_1$ in (2-201).

An infinite number of equivalent lumped-parameter networks can be used to represent $\bar{Z}_{in}$ in the vicinity of $\omega_1$. Usually, the simplest possible network is used. This equivalent network must be chosen so that its input impedance $\bar{Z}$ equals $\bar{Z}_{in}$ at $\omega_1$. Likewise, for small variations $\Delta \omega$ about $\omega_1$, the two impedances must be equal. A general procedure for specifying this equivalence is obtained by expanding the impedance functions in a power series in $\omega - \omega_1 = \Delta \omega$ about $\omega_1$ and equating these series term by term. Since $\bar{Z}_{in}$ has a pole at $\omega = \omega_1$, the Taylor expansion cannot be applied directly to $\bar{Z}_{in}$. However, it may be applied to $(\omega - \omega_1)\bar{Z}_{in}(\omega)$ to give

\[
(\omega - \omega_1)\bar{Z}_{in}(\omega) = (\omega - \omega_1) \lim_{\omega \to \omega_1} (\omega - \omega_1)\bar{Z}_{in}(\omega) + \frac{d}{d\omega} (\omega - \omega_1)\bar{Z}_{in}\bigg|_{\omega_1} (\omega - \omega_1) + \frac{1}{2} \frac{d^2}{d\omega^2} (\omega - \omega_1)\bar{Z}_{in}\bigg|_{\omega_1} (\omega - \omega_1)^2 + \cdots \quad (2-202)
\]

We now obtain

\[
\bar{Z}_{in}(\omega) = \lim_{\omega \to \omega_1} \frac{(\omega - \omega_1)\bar{Z}_{in}(\omega)}{\omega - \omega_1} + \frac{d}{d\omega} (\omega - \omega_1)\bar{Z}_{in}\bigg|_{\omega_1} + \cdots \quad (2-203)
\]

A similar expansion of $\bar{Z}$ gives (note that $\bar{Z}$ must have a pole at $\omega_1$ also)

\[
\bar{Z}(\omega) = \lim_{\omega \to \omega_1} \frac{(\omega - \omega_1)\bar{Z}_{in}(\omega)}{\omega - \omega_1} + \frac{d}{d\omega} (\omega - \omega_1)\bar{Z}\bigg|_{\omega_1} + \cdots \quad (2-204)
\]

Expansions of this type are called Laurent series expansions, and the coefficient of the $(\omega - \omega_1)^{-1}$ term is called the residue at the pole $\omega_1$. These two series must be made equal term for term up to the highest power in $\omega - \omega_1 = \Delta \omega$ required to represent $\bar{Z}_{in}$ with sufficient accuracy in the frequency range of interest. For a microwave cavity, the $Q$ is usually high, and the frequency range $\Delta \omega / \omega_1$ of interest is approximately the range between the two points where $|\bar{Z}_{in}|$ equals 0.707 of its maximum. This latter fractional
frequency band is equal to $1/Q$, and hence $\Delta \omega / \omega_1$ is so small that normally only the first terms in the expansion (2-203) would be required to represent $Z_{in}$ with sufficient accuracy in the vicinity of $\omega_1$. In the present case a simple parallel $LC$ circuit would be sufficient to represent $Z_{in}(\omega)$ around $\omega_1$.

In order to specify the values of $L$ and $C$, we must evaluate the first terms in (2-203) and (2-204). For the $LC$ circuit we have

$$Z = \frac{j \omega L}{1-\omega^2 LC}. \quad (2-205)$$

We now choose $\omega_1^2 LC = 1$ to produce a pole at $\omega_1$ for $Z$. Hence

$$Z = \frac{j \omega_1^2 L}{\omega_1^2 - \omega^2}, \quad (2-206)$$

and

$$\lim_{\omega \to \omega_1} (\omega - \omega_1) Z = \left. \frac{j \omega_1^2 L}{\omega + \omega_1} \right|_{\omega_1} = \frac{-j \omega_1^2 L}{2}. \quad (2-207)$$

Thus we have

$$Z(\omega) = \frac{-j \omega_1^2 L}{2(\omega - \omega_1)}, \quad (2-208)$$

for $\omega$ near $\omega_1$.

To evaluate the behavior of $Z_{in}$ near $\omega_1$, we can place $\omega$ equal to $\omega_1$ in the numerator in (2-199). The denominator is first expanded in a Taylor series in $\beta$ about $\beta_1$ to give

$$\bar{X}_L(\beta) + \tan \beta d \approx \bar{X}_L(\beta_1) + \tan \beta_1 d + \left( \frac{d \bar{X}_L}{d \beta} + \frac{d \tan \beta d}{d \beta} \right) |_{\beta_1} (\beta - \beta_1)$$

$$= \left( \frac{\bar{X}_L}{\beta_1} + d \sec^2 \beta_1 d \right) (\beta - \beta_1). \quad (2-209)$$

since $\bar{X}_L(\beta) = \bar{X}_{L_1} = - \tan \beta_1 d$ and $d \bar{X}_L / d \beta = (1/\beta) \bar{X}_L$. Next we expand $\beta$ in terms of $\omega$ about $\omega_1$ to give
\[ \beta \approx \beta_1 + \frac{d\beta}{d\omega} |_{\omega_1} (\omega - \omega_1). \]  
(2-210)

If we denote \( \frac{d\beta}{d\omega} \) at \( \omega_1 \) by \( \beta'_1 \), we see that \( \bar{Z}_{in} \) can be expressed as

\[ \bar{Z}_{in} = \frac{j\bar{x}_{L_1} \tan \beta_1 d}{[\bar{x}_{L_1} + \beta_1 d(1 + \tan^2 \beta_1 d)][(\beta'_1 / \beta_1)(\omega - \omega_1)].} \]  
(2-211)

Upon replacing \( \sec^2 \beta_1 d \) by \( 1 + \tan^2 \beta_1 d \). Replacing \( \tan \beta_1 d \) by \(-\bar{x}_{L_1}\) now gives

\[ \bar{Z}_{in} = -j \frac{\bar{x}_{L_1}^2}{[\bar{x}_{L_1} + \beta_1 d(1 + \bar{x}_{L_1}^2)][(\beta'_1 / \beta_1)(\omega - \omega_1)].} \]  
(2-212)

Normally, \( \bar{x}_{L_1} \ll 1 \), and since \( \beta_1 d \approx \pi \), we can make further approximations to obtain

[42] (we shall verify that \( \bar{x}_{L_1} \ll 1 \) later)

\[ \bar{Z}_{in} = -j \frac{\bar{x}_{L_1}^2}{\beta'_1 d(\omega - \omega_1)}. \]  
(2-213)

Comparison with (2-208) shows that we must choose

\[ \frac{\omega_{2L}^2}{2} = \frac{\bar{x}_{L_1}^2}{\beta'_1 d}. \]  
(2-214)

or

\[ L = \frac{2\bar{x}_{L_1}^2}{\omega_{2}\beta'_1 d}. \]  
(2-215)

The capacitance \( C \) is determined by the condition \( \omega_{2L}^2 LC = 1 \) given earlier.

Up to this point we have neglected the losses in the cavity. For a high-Q cavity these may be accounted for simply by replacing the resonant frequency \( \omega_1 \) by a complex resonant frequency \( \omega_1(1 + j/2Q) \). That is, the natural response of a lossy cavity is proportional to \( e^{-\delta t + j\omega_1 t} \), and not to \( e^{j\omega_1 t} \), where \( \delta = \omega_1 / 2Q \). The field in the cavity is, apart from some local fringing because of the presence of the aperture, a \( TE_{101} \) mode. \( Q \) of a rectangular cavity is given by (2-216) as
\[ Q = \frac{\omega W}{p_l} = \frac{2\omega W_e}{p_l} = \frac{(k_{101}a d)^3 b Z_0}{2\pi^2 R_m (2a^3 b + 2d^3 b + a^3 d + d^3 a)}, \]  

(2-216)

where \( R_m = 1/\sigma \delta \) is the resistive part of the surface impedance exhibited by the conducting wall having a conductivity \( \sigma \) and skin depth \( \delta_s = (2/\omega \mu \sigma)^{1/2} \). The dimensions of the cavity are \( a, b, \) and \( d \). For the lossy case we then have

\[ \tilde{Z}_{in} = -j \frac{X^2 L_1}{\beta'_1 d (\omega - \omega_1 - j\omega_1/2Q)}. \]  

(2-217)

At resonance \( (\omega = \omega_1) \), we now obtain a pure resistive impedance \( \tilde{R}_{in} \) given by

\[ \tilde{R}_{in} = \tilde{Z}_{in} = \frac{2 X^2 L_1 Q}{\omega_1 \beta'_1 d}. \]  

(2-218)

If we want the cavity to be matched to the waveguide at resonance, we must choose the aperture reactance \( X_{L_1} \) so that \( \tilde{R}_{in} = 1 \); that is,

\[ X_{L_1} = \left( \frac{\omega_1 \beta'_1 d}{2Q} \right)^{1/2}. \]  

(2-219)

This matched condition is referred to as critical coupling. If \( \tilde{R}_{in} \) is greater than the characteristic impedance of the input line (unity in the case of normalized impedances), the cavity is said to be overcoupled, whereas if \( \tilde{R}_{in} \) is smaller, the cavity is undercoupled. If \( \tilde{R}_{in} \) is the normalized input resistance at resonance for a parallel resonant cavity, then \( \tilde{R}_{in} \) is defined as the coupling parameter \( K \). In the case of series resonance, the coupling parameter equals the input normalized conductance at resonance.

As an example, for the rectangular cavity with \( a = b = d = 3 \text{ cm} \) we found \( f_1 = 7,070 \) MHz and \( Q = 12,700 \). For this cavity \( \beta'_1 = \frac{\omega_1}{\beta_1 c^2} = 4.7 \times 10^{11} \text{s/cm} \) and (2-219) gives \( X_{L_1} = 0.0157 \) for critical coupling. The corresponding aperture radius \( r_0 \) from (2-198) is found to be \( 0.37 \text{ cm} \). Note that \( X_{L_1} \ll 1 \), so our earlier approximation in neglecting \( X_{L_1} \)
compared with unity is justified. Also note that a solution of (2-219) for the required value of $\bar{X}_{L_1}$ to give critical coupling must, in general, be carried out simultaneously with the solution of (2-200) for the resonant frequency $\omega_1$. However, for a high-Q cavity, $\omega_1$ may be approximated with negligible error by the frequency corresponding to $\beta d = \pi$ in (2-219). This was done in the above calculation.

For the lossy cavity the equivalent circuit must include a resistance $\bar{R}_{in}$ in parallel with $L$ and $C$ as illustrated in Figure 2.11a. It may readily be verified that the input impedance $\bar{Z}$ now becomes, for $\omega$ near $\omega_1$,

$$\bar{Z} = -j \frac{\omega_1^2 L}{2(\omega - \omega_1 - j\omega_1/2Q)}.$$  (2-220)

Where $Q = \bar{R}_{in}/\omega_1 L$.

Since the cavity is coupled to an input waveguide, the cavity terminals are loaded by an impedance equal to the impedance seen looking toward the generator from the aperture plane. If the generator is matched to the waveguide, a normalized resistance of unity is connected across the cavity terminals, as in Figure 2.11b. The external $Q_e$ is given by

$$Q_e = \frac{1}{\omega_1 L}.$$  (2-221)

And the loaded $Q_L$ by

$$Q_L = \left(\frac{1}{Q} + \frac{1}{Q_e}\right)^{-1} = \frac{\bar{R}_{in}}{(1 + \bar{R}_{in})\omega_1 L}.$$  (2-222)
The loaded and unloaded quality factors are related as follows

\[ \frac{1}{Q_L} = \frac{1}{Q} + \frac{1}{Q_e} = \frac{(1+\bar{R}_{in})\omega_1 L}{\bar{R}_{in}} = \omega_1 L + \frac{\omega_1 L}{\bar{R}_{in}} = \frac{1}{q}(1 + K) , \]

or \[ Q = (1 + K)Q_L . \] (2-223)

In general, the coupling parameter \( K \) may be defined as

\[ K = \frac{Q}{Q_e} . \] (2-224)

For a parallel resonant circuit this is seen to give

\[ K = \frac{\bar{R}_{in}}{\omega_1 L} . \] (2-225)

which agrees with the earlier definition. Likewise, for a series resonant circuit with normalized input conductance \( \bar{G}_{in} \), the unloaded and loaded \( Q \)'s are given by \[ Q = \omega_1 L\bar{G}_{in} , \ Q_e = \omega_1 L , \] and hence \( K = \bar{G}_{in} = Q/Q_e \) again. The coupling parameter is a measure of the degree of coupling between the cavity and the input waveguide or transmission line.

The external \( Q , Q_e \), is sometimes called the radiation \( Q \). The reason for this is that the cavity may be considered to radiate power through the aperture into the input waveguide. This power loss by radiation through the aperture is equal to the power lost in the normalized unit resistance connected across the resonator terminals in the equivalent circuit illustrated in Figure 2.11b.

For the rectangular cavity under discussion, the next–higher frequency at which a resonance occurs corresponds to a series-type resonance at which \( |\bar{Z}_{in}| \) is a minimum. In the loss-free case, \( \bar{Z}_{in} = 0 \), and from (2-199) this is seen to correspond to \( \tan \beta d = 0 \), or \( \beta d = \pi \). At a series resonance, \( \bar{V}_{in} \) has a pole, and consequently an analysis similar to that
presented for $\tilde{Z}_{in}$ is applied to $\tilde{Y}_{in}$. It is readily found that in the vicinity of $\omega = \omega_2$, where $\omega_2$ is the value of $\omega$ that makes $\beta d = \pi$, the input admittance $\tilde{Y}_{in}$ would be

$$\tilde{Y}_{in} = -j \frac{1}{\beta' \frac{\omega - \omega_2 - j \omega_2 / 2Q}{\omega}}.$$  

(2-226)

In this case the aperture has no effect, as would be expected, because the standing-wave pattern in the cavity is now such that the transverse electric field is zero at the aperture plane. The input admittance near $\omega_2$ is just that of a short-circuited guide near a half guide wavelength long. This resonance is not of any practical interest since it corresponds to a very loosely coupled cavity.

### 2.8.3 Loop-Coupled Cavity

Figure 2.12 illustrates a cavity that is coupled to a coaxial line by means of a small loop. Since the loop is very small, the current in the loop can be considered to be constant. Any mode in the cavity that has a magnetic field with flux lines that thread through the loop will be coupled by the loop. However, at any particular frequency $\omega$, only that mode which is resonant at this frequency will be excited with appreciable amplitude. The fields excited in the cavity by the current $I$ flowing in the loop can be found by solving for a vector potential arising from the current $I$ [42].

![Coaxial line and cavity](image)

Figure 2.12 Loop-coupled cavity.

From the vector potential the magnetic field, and hence the flux passing through the loop,
may be found. For a unit current let the magnetic flux of the \( n \)th mode that threads through the loop be \( \psi_n \). This is then equal to the mutual inductance \( M_n \) between the coupling loop and the \( n \)th mode. Each mode presents an impedance equivalent to that of a series \( RLC \) circuit to the coupling loop. Thus a suitable equivalent circuit is an infinite number of series \( RLC \) circuits coupled by mutual inductance to the input coaxial line, as illustrated in Figure 2.13.

The input impedance is thus of the form (2-227) [43]

\[
Z_{\text{in}} = j\omega L_0 + j \sum_{n=1}^{\infty} \frac{\omega^3 M_n^2 C_n}{1 - \omega^2 L_n C_n + j\omega C_n R_n},
\]

where \( L_0 \) is the self-inductance of the coupling loop. If we define the resonance frequencies \( \omega_0 \) by \( \omega_n^2 L_n C_n = 1 \) and the unloaded \( Q \) of the \( n \)th mode by \( Q_n = \frac{\omega_n L_n}{R_n} \), we can rewrite (2-227) as

\[
Z_{\text{in}} = j\omega L_0 + j \sum_{n=1}^{\infty} \frac{\omega^3 M_n^2}{L_n (\omega_n^2 - \omega^2 + j\omega \omega_n/Q_n)} .
\]

If \( \omega \approx \omega_n \), then all terms in the series in (2-228) except the \( n \)th term are small. Thus, in the vicinity of the \( n \)th resonance,

\[
Z_{\text{in}} = j\omega L_0 + j \frac{j\omega^3 M_n^2}{L_n (\omega_n^2 - \omega^2 + j\omega \omega_n/Q_n)} \approx j\omega L_0 - j \frac{\omega_n^2 M_n^2}{2L_n (\omega - \omega_n)^2 \omega_n/Q_n} .
\]

The equivalent circuit now reduces to a single \( LCR \) series circuit mutually coupled to the input line. For efficient excitation of a given mode, the loop should be located at a point where this mode provides a maximum flux linkage.

The preceding results represent a formal solution to the loop-coupled cavity. In order to specify the circuit parameters, the boundary-value problem for the fields excited in a cavity by a loop current must be solved. Also the \( Q \)'s of the various cavity modes must be
determined. For simple cavity shapes these calculations can be carried out with reasonable accuracy. However, they are too lengthy to include here.

2.8.4 Large Aperture Coupling at Microwave Frequencies

Because of the importance of apertures in many different microwave devices, considerable theoretical and experimental work has been done in the past on this type of circuit element.

For example, circular and rectangular transverse irises in waveguide have been analyzed thoroughly by various investigators [33], [43]. Apertures of shapes other than these have not thus far had accurate solutions that take full account of the proximity of the aperture to the waveguide walls, or that are valid near the resonant frequency of the aperture. Bethe has worked out a general approach to the problem that can be used with an aperture of any shape in an infinitely thin conducting wall between any two regions if the aperture is small compared to the wavelength and to the distance to the nearest sharp bend of the wall or any other perturbation [44]. Specifically, the resonant frequency of the aperture should not be less than three times the operating frequency if good accuracy is to be obtained.

In many microwave devices, such as filters, antennas and broad-banding elements, apertures are used at or near resonance, and, hence Bethe’s coupling formulas are not applicable. In this section a frequency-correction factor will be added to Bethe’s coupling formula for apertures in a transverse diaphragm in a rectangular waveguide. Also, an approximate correction for wall thickness will be presented. Although coupling between other regions has not been investigated, the same principles should be applicable.
Figure 2.13 Equivalent circuit for a loop-coupled cavity.
2.8.4.1 The High-Frequency Response of a Large Aperture

The effect of an infinitely thin perforated transverse diaphragm on the fundamental mode of a waveguide may be computed from an equivalent circuit in which the diaphragm is represented by a two-terminal impedance shunted across a two-conductor transmission line that is assumed to carry only the fundamental mode of the waveguide. This impedance is essentially lossless, and therefore must be of the form specified by Foster’s reactance theorem [45], [46]. Foster’s theorem, which holds for any lossless, non-active, linear, two-terminal network, may be expressed in the following mathematical form

\[ X(f) = -\frac{1}{B(f)} = \sum_{m=0}^{n} \frac{p_m f}{1 - f^2 / f_m^2} + q f - \frac{r}{f} \quad 0 < f_0 < f_1 < \cdots < f_n \], \quad (2-230)

where \( X(f) \) is the reactance and \( B(f) \) the susceptance of the network; \( f \) is the frequency; \( f_m \) the frequencies of the poles of \( X(f) \); and \( p_m, q \) and \( r \) are positive, real constants. For a lumped-constant network, \( n \) is finite; while for a distributed constant network, such as a diaphragm in a waveguide, \( n \) is infinite, and an essential singularity of the function occurs at \( f = \infty \).

The reactance function for a small aperture in an infinitely thin conducting diaphragm in a transverse plane of a rectangular waveguide (\( TE_{10} \) mode) is [44]

\[ X = -\frac{1}{B} = \frac{4\pi M Z_0}{ab \lambda_0} \], \quad (2-231)

Where \( M \) is the magnetic polarizability of the aperture, \( Z_0 \) the characteristic impedance of the waveguide, \( a \) and \( b \) the width and height of the waveguide cross section, and \( \lambda_0 \) the guide wavelength. Formulas for \( M \) are known for circular and elliptical apertures [44, 47] and accurate measured values are available for the shapes of Figure 2.14 [48]. Since \( Z_0 \) is
proportional to $\lambda_g/\lambda$, it follows that $X \propto f$, and hence (2-231), assumes the aperture to have the reactance of a constant inductance. Comparison with (2-230) shows that the small-aperture theory assumes $q$ finite, $r = 0$, and $p_m = 0$, for all values of $n$. This of course is not actually the case, since an aperture has an unlimited number of resonances; however, it is a very good approximation for $f \ll f_0$.

Now consider the reactance function that is one step more complicated than a straight-line function, but that reduces to (2-231) for $f \to 0$

$$\frac{X}{Z_0} = -\frac{Y_0}{B} = \frac{4\pi M}{ab\lambda_g(1-f^2/f_0^2)}.$$ \hspace{1cm} (2-232)

Another simple special case is that for which the poles occur at $f = (2m + 1) f_0$ and zeros at $f = 2mf_0$, where $m$ takes on all integer values from $-\infty$ to $\infty$

$$\frac{X}{Z_0} = \frac{4\pi M}{ab\lambda_g} \left( 2f_0 \frac{f\tan \frac{\pi f}{2f_0}}{\pi f} \right).$$ \hspace{1cm} (2-233)

It can be seen that an infinite number of other reactance functions that conform with (2-230) and that reduce to (2-231) are possible. Nevertheless, it has been found through experimental investigation of many aperture shapes that the single-pole relation of (2-232) gives very good agreement in all cases with the measured values, and is in most cases superior to (2-233) [45], [46]. Thus, (2-233) is, in general, recommended up to and somewhat beyond the first resonance of the aperture.

### 2.8.4.2 The Resonant Frequency of an Aperture

In order to use (2-232), it is necessary to know the resonant frequency of the aperture. This may be determined experimentally, or by certain empirical relations that will now be given.
The following empirical relationship for the resonant length $l$ of a rectangular aperture of width $w$, centered in a transverse plane of a rectangular waveguide, was originally suggested by Slater and then developed by Nedlin [46]

$$\lambda_0 \frac{2}{\sqrt{1 + \left(\frac{2aw}{b\lambda_{g0}}\right)^2}}.$$  \hspace{1cm} (2-234)

Equation (2-234) is generally accurate to within a few percent. It is the equation used to design a coupling slot for the experiments described in Chapter 3. For $w/l$ small, (2-234) reduces to $l = \lambda_0/2$. Hence, the resonant frequency of a narrow rectangular aperture is approximately equal to the cutoff frequency of a waveguide having the same cross-sectional shape as the aperture. This correspondence has been found to be valid for several other aperture shapes whose width-to-length ratio is small, and it is, therefore, suggested as a general approximation. For example, consider the rounded slot of Figure 2.14a. If the cross section of the corresponding waveguide is narrow enough so that the electric field is negligible in the semicircular ends, the cutoff frequency should be the same as that of a rectangular waveguide having the same height $w$ and the same cross-sectional area. Thus, the resonant length of a narrow rounded slot is

$$l = \lambda_0 \frac{2}{2} + 0.273w.$$  \hspace{1cm} (2-235)

This was found to agree within one percent for five apertures having $w/l$ under 0.11, and within 5 percent for $w/l = 0.2$. Approximate formulas for the cutoff wavelength of a number of other waveguide cross sections have been or can be obtained. For example, a theoretically computed graph for a "dumbbell" cross section has been given by Bethe [44] and for a "ridged" cross section by Cohn [49].
If theoretical or experimental data are not available for a particular aperture shape, a rough guess can usually be made that will yield good results if the aperture is not used too near resonance. For example, in the case of apertures (d) and (e) of Figure 2.14, (2-235) may be used with fair accuracy if \( w/l \) is small.

### 2.8.4.3 The Resonant Q of an Iris

The Q of a resonant iris loaded by matched waveguide terminations may be defined by

\[
Q = \frac{f_0 d(B/Y_0)}{2} \left. \frac{df}{f} \right|_{f=f_0} .
\]  

(2-236)

Substitution of (2-232) in (2-236) gives

\[
Q = \frac{ab\lambda g_0}{4\pi M} .
\]  

(2-237)

Because of the effect of the higher resonant frequencies that were neglected in the derivation of (2-237), and because of the finite thickness of a practical diaphragm, the actual Q of an aperture would be somewhat greater than the value predicted by (2-237).

### 2.8.4.4 Effect of the Wall Thickness

The attenuation \( \alpha_0 \) introduced by an aperture in an infinitely thin transverse conducting diaphragm in a waveguide is given by

\[
\alpha_0 = 10 log_{10} \left[ \left( \frac{B}{2Y_0} \right)^2 + 1 \right] dB .
\]  

(2-238)

If the diaphragm has finite thickness and if the frequency is below resonance, the attenuation will be greater than \( \alpha_0 \). For very large thickness it is obvious that the increase
Figure 2.14 Additional apertures for which accurate magnetic polarizability data are available.

Attenuation will approach asymptotically the attenuation of the principal mode in a waveguide whose cross section is the shape and size of the aperture, and whose length is equal to the thickness of the diaphragm [33]. Hence the following formula for the added attenuation is suggested

\[ \alpha_t = \frac{54.6tA}{\lambda_c} \sqrt{1 - \left(\frac{\lambda_c}{\lambda}\right)^2} \quad dB. \]  

(2-239)

This is the attenuation formula for a waveguide with an additional factor, A. This factor is a function of the thickness t, and approaches unity for t large; \( \lambda_c \) is the cutoff wavelength of the waveguide. The total attenuation of the diaphragm is

\[ \alpha = \alpha_0 + \alpha_t. \]  

(2-240)
The experimental investigations showed that in all cases tested, $A$ is approximately three for $t < 0.02l$, and decreases slowly with increasing $t$. For the smaller values of $w/l$, $A$ tends to be somewhat greater than three, while for the larger values of $w/l$, $A$ tends to be somewhat less than three. In the case of a circular aperture, $A$ appears to be only slightly greater than one [33]. The effect of thickness on resonant frequency has been found to be small for aperture ($a$) of Figure 2.14. For aperture $c$, however, this effect is large.

2.9 Choke Design

A choke joint may be made in much the same manner as in coaxial line. A gap in the form of a vertical slit in the narrow face of the waveguide has only a small effect, since the currents in the wall are in the direction of the length of slit. A gap in the broad face of the waveguide would produce, however, a large disturbance. A short-circuited stub line one-half wavelength long, which is broken at the quarter-wavelength point, would form a good choke joint. A practical design of a choke connector for rectangular waveguide is shown in Figure 2.15. The circular choke groove is easy to manufacture, and the high- and low-impedance stub sections are incorporated in the design. The proper diameter of the groove must, of course, be determined experimentally; the proper depth of the groove is very close to one-quarter of a free-space wavelength [47].

Stub tuners can also be constructed in waveguide through the use of series junctions. The stub can be waveguides of either round or rectangular cross section. Such a stub introduces a series reactance in the line rather than a shunt susceptance. The application
of the duality principle, however, allows the matching formulas and the tuning range that apply for the shunt circuit to be readily converted to the series case.

Plungers are usually designed with choke joints as indicated in Figure 2.15a. Some form of adjustable short circuit or plunger can be used to vary the length of a stub line. Another design that is very similar employs three quarter-wavelength sections as shown in Figure 2.15b. As a typical example, if the characteristic impedances of the choke sections, which are proportional to the waveguide diameter for cylindrical waveguides, are those designated in the figure; the input impedance of the plunger (Z) can be made very small. For instance, the low impedance section \((Z_1/Z_0)\) can be made small, perhaps 0.1, and the high impedance section \((Z_2/Z_0)\) can be 0.5, \(Z\) can easily be as small as \(4 \times 10^{-4}\), and the power loss is therefore about 70 dB. These values are adequately small for nearly every application.

\[
Z = Z_0 \frac{(Z_1/Z_0)^4}{(Z_2/Z_0)^2} \tag{2-241}
\]

Figure 2.15 Chokes (a) T-shape choke, (b) Stepped impedance choke. \(Z_1\), \(Z_2\) and \(Z\) are respectively low impedance, high impedance and input impedance.
2.10 References


Chapter 3- Simulations and Experimental Results of Electron Density Measurement using Microwave Perturbation Method

A cylindrical DBD plasma under five different pressures is generated in an evacuated glass tube. This plasma volume is located at the center of a rectangular copper waveguide cavity where the electric field is the maximum for the first mode and the magnetic field is very close to zero. The microwave perturbation method is used to measure electron density and plasma frequency for these five pressures. Simulations by a commercial microwave simulator are comparable to the experimental results.

Plasma actuators for aerodynamic applications are receiving significant research attention [1, 2] and it is necessary to know the effects of this plasma on the vehicle radar cross section (RCS). Plasma has a capability that its refractive index can be controlled by changing parameters such as the electron density profile, plasma frequency or collision rate so the RCS becomes controllable. However controlling the aforementioned parameters practically in order to decrease RCS is not as simple as theoretical simulations. Many numerical methods such as finite-difference time-domain method (FDTD) [3], method of moments (MoM) [4], finite element method (FEM) [5] and impedance method [6] are used to simulate electromagnetic wave interaction with a plasma layer. Most of these simulations are used for particular conditions like collision-less plasma to show RCS reduction [6], while collision-less plasma can only be generated in low density plasma in a low pressure situation or in the sheath region surrounding a re-entry vehicle.

Plasma is a state of matter in which the negative ions or electrons and positive ions are available at the same time. In fact plasma is a quasi-neutral mix of electrons, ions and
neutral particles. Not all plasmas are fully ionized and the percentage of ionization depends on the application for which the plasma is generated. These applications vary widely from plasma TV to plasma antennas [7] and plasma stealth technology [8, 9].

Plasmas can be generated at different temperatures from close to zero Kelvin to higher than 10 GK, so the plasma can be classified as hot or cold. In hot plasma there is a thermodynamic state which is called Local Thermodynamic Equilibrium (LTE). This state is characterized by the property that all particle concentrations are only a function of temperature. This plasma is sometimes called thermal plasma. Cold plasmas, on the other hand, are characterized by the property that the energy is selectively fed to the electrons, so the electron temperatures can be considerably higher than the heavy particles in the plasma. This kind of plasma is called the non-equilibrium or non-LTE plasma and has the typical properties such as electrical conductivity, light emission and chemical activity. Cold plasma is the type of plasma used for many applications including plasma stealth [10] and dielectric barrier discharge plasma is also in this category of non-thermal or cold plasma [11].

Plasma can be magnetized or un-magnetized depending on the application for which it is generated. Magnetized plasma is achieved by applying a DC or AC magnetic field. If the electron enters the magnetic field at an angle to the field direction the resulting path of the electron (or indeed any charged particle) will be helical. It acts to confine electrons in the region of magnetic field, so there is an increase in the collision rate which is beneficial for some applications like plasma stealth and RCS reduction [6]. An electromagnetic wave in plasma has a magnetic field in which electrons oscillate. In un-magnetized plasma these electron waves or oscillations are related to the incident wave,
but in magnetized plasma the electron waves have four modes. Modes O (Ordinary) and X (eXtraordinary) which are perpendicular to the external magnetic field and R (Right-hand circularly polarized) and L (Left-hand circularly polarized) which are parallel to the field. For O-mode the dispersion relation and cut-off frequency are the same as unmagnetized plasma shown in equation (3-1), but for the other three modes the dispersion relation is more complicated. In plasma actuators we will restrict our investigation to the unmagnetized case.

Cold plasma acts as a high-pass filter in that electromagnetic waves above a particular frequency, known as the plasma frequency \( \omega_p \), are transmitted while electromagnetic waves below the plasma frequency are reflected. In a cold plasma approximation, the plasma frequency can be related to the wave number by using (2-134) for collision-less plasma as in equation (3-1). In this equation \( \omega \) is the incident wave frequency, \( c \) is the speed of light in a vacuum and \( k \) is the wave number.

\[
\omega^2 = \omega_p^2 + c^2 k^2 .
\]  

(3-1)

Where \( \omega_p \) defines the cut-off frequency in (rad/s) at which the plasma changes from reflective to transmissive. The electron density \( n_e \) is a parameter that defines the plasma frequency. They are related by equation (3-2) [12] as it is in (2-18).

\[
\omega_p = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}} \Rightarrow f_p \cong 9 \sqrt{n_e} .
\]  

(3-2)

Here \( e \) and \( m_e \) are the electron charge and mass respectively, \( \varepsilon_0 \) is the permittivity of vacuum, and \( \omega_p = 2\pi f_p \). The most widely used methods to measure electron density and plasma frequency are reflectometry [13-15] and interferometry [16], both based on the same principles.
3.1 Simulations

The method used for this research is the microwave perturbation method at S-Band as it provides very high sensitivity to these weakly ionized plasmas. Figure 3.1 shows the setup for simulation and experimental measurements. The S-band waveguide has dimensions of 34.04 mm by 72.14 mm. The wall thickness is 2 mm and the material is copper with $\sigma = 5.96 \times 10^7 S/m$. 
Figure 3.1 (a) The rectangular cavity, waveguide and cylindrical ears used for simulation and experimental measurements. (b) Chokes inside the copper ears, high voltage leads and plasma region at the center. (c) Side view of the Chokes using a cut plane.

The perturbation method works when the dimensions of the sample are much smaller than the dimensions of the cavity. To measure permittivity the sample must be located inside the cavity where the electric field is the maximum and if to measure permeability the sample would be placed where the magnetic field is the maximum. For the first mode of the cavity ($E_{101}$) the electric field is maximum at the center of the cavity. As shown in Figure 3.2, the electric field in the $z$ direction dominates and the magnetic fields are at least 60 dB below the maximum. The difference between $E_z$ and $E_x$ at the first resonant mode is about 160 dB. However it seems a big number for dynamic range, but it is acceptable when we know that CST accuracy limit is -200 dB [17]. The length of the cavity is 12.5 cm and a slot aperture is used for coupling between the waveguide and the cavity. The dimensions of the orifice (coupling slot) are 50 mm by 2 mm. As depicted in Figure 3.1b the blue cylinder in the middle of the cavity shows the plasma region which is 10 mm in length with a radius of 5 mm corresponding to the size of the electrodes on
both sides. These dimensions results in a sample to cavity volume ratio of $\frac{V_1}{V_0} = 0.00256$.

In general the perturbation formulae [18] can be written as equations (2-150) and (2-167).

![Probe Value in V/m [Magnitude in dB]](a)

![Probe Value in A/m [Magnitude in dB]](b)

Figure 3.2 (a) Electric field at the center of the cavity along each axis. (b) Magnetic fields components at the center of the cavity. The bold vertical line shows the resonance frequency.

If $E_0, D_0, H_0, B_0$ and $E_1, D_1, H_1, B_1$ are respectively unperturbed and perturbed electric fields, electric displacement fields, magnetic field strength and magnetic field flux densities, then the same procedure used to achieve Equation (2-170) for the spherical
dielectric can be used for a cylindrical dielectric by knowing the electric field in the
cylinder is \( E_1 = \frac{2}{\varepsilon_{r+1}} E_0 \). Since at the middle of the cavity the electric field is maximum,
the magnetic field becomes very small and \( \varepsilon_r \approx 1 \), these equations can be simplified to
(3-3) and (3-4) when the sample has a cylindrical form.

\[
\frac{\delta \omega}{\omega_0} = -A(\varepsilon - 1), \tag{3-3}
\]

\[
\frac{\delta \Omega}{\omega_0} = -A(\varepsilon' - j \varepsilon'' - 1). \tag{3-4}
\]

Where \( A \) is a constant which depends on the geometrical shape and the volume ratio of
the sample and the cavity. While \( \varepsilon' \) and \( \varepsilon'' \) are the real and imaginary parts of the
permittivity. For a cubic cavity and cylindrical sample it would result in \( A = \frac{2v_1}{v_0} \), so the
permittivity of the sample can be found by rearranging these equations as shown in (3-5)
and (3-6).

\[
\varepsilon' = \frac{v_0(f_0 - f_1)}{2v_1f_1} + 1, \tag{3-5}
\]

\[
\varepsilon'' = \frac{v_0}{4v_1} \left( \frac{1}{Q_1} - \frac{1}{Q_0} \right). \tag{3-6}
\]

where \( f_1 \) is the resonance frequency after inserting the sample.

The cylindrical ears have 14 mm inner diameter which makes a high impedance for S
band, if operated as cylindrical waveguide. Since two copper wires are used to feed the
electrodes in order to generate plasma in the region between the electrodes, chokes are
needed to prevent microwave leakage through these coaxial transmission lines and to
maintain a high cavity Q. As shown in Figure 3.1(b) the chokes are designed as stepped
impedance structures [19]. Chokes are designed based on the Equation (2-241). There are
four sections on the right and left sides for the stepped impedance. The diameter of the
wire lead or high impedance segment \( (Z_2) \) is 1.34 mm and the length is 35 mm. The
diameter of the low impedance segments \( (Z_1) \) is 6.4 mm and their length is 17 mm. The
input impedance seen inside the cavity at the input of cylindrical ear would be \( Z_{\text{in}} =
Z_0 \left( \frac{Z_2}{Z_1} \right)^4 \) where, \( Z_0 \) is the free space impedance. Therefore, the input impedance for this
choke would be 352 kΩ. This impedance is a high impedance compared to waveguide
impedance for the first mode \( (E_{10}) \) at 2.845 GHz which is 552Ω. For this impedance ratio
the power leakage would be less than 0.0015dB which provides enough accuracy for the
experimental tests.

The plasma can be modeled as a dielectric material using a specific Drude model. The
real and imaginary parts of the plasma permittivity based on the Lorentz-Drude model are
given in (3-7) as a result of (2-120) when cyclotron frequency is zero [20-21].

\[
\varepsilon_r = 1 - \frac{\omega_p^2}{\omega(\omega + jv)} = 1 - \frac{\omega_p^2}{\omega^2 + v^2} + \frac{\nu}{\omega} \frac{\omega_p^2}{\omega^2 + v^2} j = \varepsilon'_r + j\varepsilon''_r. \tag{3-7}
\]

Where \( v \) is electron-neutral collision rate (rad/s).

An S\(_{21}\) simulation of the transmission line coupled cavity is used to determine the
quality factor and resonance frequency of the resonator. S\(_{21}\) Terminals are on two ends of
the waveguide. These parameters are shown in Figure 3.3 where the quality factor and the
resonance frequency before plasma activation are 1479 and 2.845 GHz respectively.
Figure 3.3 Simulated response of the transmission line coupled cavity. (a) $S_{21}$ of the cavity when plasma is off. (b) $S_{21}$ zoomed in for 3dB bandwidth.

For simulations, the same size structure and materials of the experimental system are used and a commercial simulator (CST-MWS) is employed. When this simulation is run for 70 different plasma frequencies and collision rates we achieve a general trend revealed on the 2D contours of Figure 3.4. These 2D contours are based on cavity resonance frequency shift. It shows the theoretical dependency or sensitivity of collision rate, plasma frequency and resonance frequency shift. For instance, if the collision rate increases to 300 GHz, the frequency shift becomes zero for plasma frequencies less than 35 MHz or the software does not have enough sensitivity for this range of changes. This plot allows us to quickly determine if a particular plasma will yield a measurable response in our experimental system. Moreover, the experimental tests are done for five different pressures as indicated in Table I. The results of these measurements are mapped on these 2D contours indicated by black dots and they show good agreement with theoretical calculations. For these contours the frequency shift ($\delta f$) is in kHz, plasma frequency ($f_p = \omega_p/2\pi$) is in Hz, and collision rate ($\nu$) is in Hz.
Figure 3.4 2D contour resulted from simulation showing cavity resonance frequency shift (kHz) for different plasma frequency ($f_p$) and collision rate ($\nu$). Filled circles show the five experimental points for five different pressures and $f_p - \nu$ pairs (Table 3.1).

### 3.2 Experimental Results

Figure 3.5 shows the setup for experimental measurements while Figure 3.5b is the block diagram schematic of Figure 3.5a. The network analyser is an Agilent PNA N5242A and the oscilloscope a LeCroy 424. The signal generator is a WAVETEK FG2C and the amplifier employed is a YAMAHA P7000S. To generate high voltage for making plasma in air, the output voltage of the amplifier is applied to a transformer with primary to secondary ratio of 1:25.

The plasma excitation voltage is applied to the test fixture and is monitored by the oscilloscope through a high voltage probe (Fluke 80K-6). The plasma current is also monitored by adding a resistor in series and reading the voltage on the oscilloscope. The vacuum pump is a two-stage model that can generate a minimum absolute pressure of 0.3
Pascal. A vacuum controller (J-KEM PSV-3) is connected between the vacuum pump and the device under test to stabilize the pressure at the desired value.

The DBD plasma formed in the test fixture from the audio frequency high voltage excitation is forming and collapsing with each half cycle of the excitation. This pulsing time-domain behaviour of the plasma can be measured on average by using a slowly responding system or can be examined at various stages by synchronizing to the excitation voltage and delaying the measurement trigger as needed. In this work we are

Figure 3.5 Experimental setup including test fixture, vacuum pump, regulator and lines, audio part, RF lines, and PNA.
measuring at the peak voltage point corresponding to the most intense plasma. The measurements are conducted in the time-domain but for 250 different frequencies, so the number of sweep points is 250 and the sweep type is continuous wave (CW). The PNA RF bandwidth is the same 3 dB bandwidth as the resonating cavity which is 2.5 MHz. The IF bandwidth of the PNA is set to 600 kHz to balance between the noise and the signal received. For each frequency the peak value is measured 12 times and then averaged, so a sweep consists of 3000 measurements. A program was written using visual basic and predefined objects of the PNA to automate the acquisition. The PNA acquisition is triggered from the oscilloscope to match the plasma peak. This measurement was repeated for five different pressures from 5 to 45 Torr in steps of 10 Torr. For pressures more than 50 Torr we could not measure any frequency shift in the cavity resonance frequency. In order to measure for higher pressures, a cavity with higher quality factor would be needed. The measurement results are shown in Figure 3.6.
Figure 3.6 Line with circle marker shows $S_{21}$ of the cavity under vacuum when plasma is off and blue curve is $S_{21}$ when plasma is on under low pressure conditions. (a) 5 Torr. (b) 15 Torr. (c) 25 Torr. (d) 35 Torr. (e) 45 Torr.

The measured results of Figure 3.6 confirm the simulated results presented in Figure 3.4 where by increasing the pressure, the collision rate increases while $f_p$ and the frequency shift ($\delta f$) decrease.
Figure 3.7 Graphs of measured and simulated values for 5 different pressures. Blue line shows the measured values and red one with circle markers shows simulated one. (a) Resonance frequency shift. (b) Quality factor change. (c) Plasma frequency. (d) Collision rate. (e) Real Part of Permittivity. (f) Imaginary Part of Permittivity. (g) Electron Density.

Figure 3.7(a) shows good agreement between measured and simulated frequency shift as the pressure is changed. In Figure 3.7(b) the measured Q is some 16% below that simulated. This slightly lower measured Q is expected as all loss mechanisms including air humidity are not accounted for the model or there is slight discrepancy between simulation and real aperture dimensions. Figures 3.7(c), (d) show the measured plasma frequency and collision rate, but these two parameters are not based on a direct measurement. In fact, the resonance frequency shifts and quality factor change are measured and then based on these measurements the real part and imaginary part of the permittivity are calculated using (3-5) and (3-6), and for the last step the plasma frequencies and collision rates for different pressures are calculated using (3-7). Another parameter that affects the electron density is the electric field intensity or the voltage applied for plasma generation. In this work, the experiments are performed at a voltage of 3000 V peak-to-peak and a frequency of 5 kHz. The current peak for this structure and voltage at the pressure of 25 Torr is 50 𝜇A. Figure 3.8 shows the voltage and current waveforms. The spikes show the time when the plasma is on.
Figure 3.8 Voltage and current waveforms for a pressure of 25 Torr.

A summary of plasma properties determined from our perturbation method is revealed in Table 3.1. The first column shows the equivalent elevation associated with the experimental air pressure.

<table>
<thead>
<tr>
<th>Elevation (m)</th>
<th>Pressure (Torr)</th>
<th>Frequency Shift(Δf) Hz</th>
<th>Plasma frequency ($f_p$) MHz</th>
<th>Collision Rate ($\nu$) GHz</th>
<th>Electron Density($n_e$) $10^14/m^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>35166</td>
<td>5</td>
<td>-113000</td>
<td>255</td>
<td>0.57</td>
<td>8.05</td>
</tr>
<tr>
<td>27490</td>
<td>15</td>
<td>-83000</td>
<td>216</td>
<td>0.92</td>
<td>6.28</td>
</tr>
<tr>
<td>23900</td>
<td>25</td>
<td>-51000</td>
<td>182</td>
<td>1.16</td>
<td>4.08</td>
</tr>
<tr>
<td>21554</td>
<td>35</td>
<td>-31000</td>
<td>148</td>
<td>1.46</td>
<td>2.69</td>
</tr>
<tr>
<td>19788</td>
<td>45</td>
<td>-18000</td>
<td>123</td>
<td>2.04</td>
<td>1.87</td>
</tr>
</tbody>
</table>

Plasma actuators operating in the lower atmosphere produce a plasma with a very high collision rate and low electron density such that its dielectric properties are not discernable from that of air with our very sensitive experimental procedure. At lower pressures corresponding to higher elevations we obtained measurable results that agree well with previous plasma models.
3.3 References


Chapter 4 - The Effect of Dielectric Barrier Discharge Plasma Actuators on Electromagnetics

4.1 Introduction

Potential benefits of electro-hydrodynamic (EHD) devices are very large spanning several applications which have prompted significant interest and investment worldwide [1]-[4]. Nevertheless, because of the complexity of the physical phenomena involved most research is limited to controlled laboratory environments while focusing on the understanding of key parameters affecting performance of those devices. There remain several issues to be overcome before the technology can be integrated into practical applications [5], [6]. For instance, the effects of operating conditions on DBD actuator performance, electrical power requirements and suitability to high-speed regimes, in case of aeronautical and aerospace, are some of the parameters that shall be addressed before using such devices in practical systems [7], [8].

Associated with the enhancements expected from the use of plasma in terms of aerodynamic performance there will also be changes in the radar cross section (RCS) performance of the vehicle [9]. These RCS changes may take the form of reduced cross section through plasma absorption or increased cross section through a change in the electrical shape of the target or the creation of a plasma plume [10].

The RCS of any object may be thought of as the projected area of an equivalent isotropic reflector. The equivalent reflector returns the same power per unit solid angle. Thus the scattering ability of a target is expressed in units of area, normally square
meters. This scattering cross-section is highly dependent upon the target shape, materials and viewing angle [9]. In this section we investigate the effect of a plasma slab on the RCS of a conductive sheet. The RCS is simulated and measured to verify the validity of the model used for simulations. Moreover, we show that the DBD actuator generates an extra reflection because of Bragg diffraction. This phenomenon is not previously reported for DBD actuators in the literature.

4.2 RCS Simulations from conductive flat plate and DBD actuators

In order to simulate and measure the effect of a plasma slab on the RCS of a conductive layer, an experimental setup was used as shown in Figure 4.1. This structure allows the generation of plasma that stands by itself in a tube. A conductive layer is installed behind the tube and is used to reflect the RF signal back to the tube. Plasma is generated in a cylindrical glass tube in which pressure can be controlled. The outer diameter of the tube is 51 mm and the inner diameter is 47 mm. Copper tape of width 25 mm and length 100 mm is used as electrodes on the exterior of the glass tube. A (0.96 - 1.45 GHz) WR770 waveguide with aperture dimensions of 196 mm by 98 mm is used as the transmitting and receiving antenna to generate a RF signal polarized perpendicular to the electrode length. The distance between the waveguide and the glass tube is 70 mm and between the conductive layer and tube is 120 mm. Using this structure, a plasma region can be generated in order to measure its effectiveness for RCS suppression. The same experiment was repeated using a WR90 waveguide for x-band (9-11 GHz).
4.2.1 Validation of Modelling

Figure 4.2 shows the comparison between simulated and measured values of the reflection from a plasma layer of thickness 25 mm, corresponding to the electrode width. The graph shows the difference in S11 for the case of the plasma present less the case of the plasma absent, with all other elements remaining constant. First, the S11 of the copper plate is simulated and measured when the plasma is off. Then, the plasma is turned on and the simulations and measurements are performed for two different pressures (15 Torr and 35 Torr). The S11 for both pressures is recorded when the plasma is off and on. The values of S11 for the case with no plasma are subtracted from the ones with the plasma layer on, so the S11 differences for both gas pressures are recorded. The plasma parameters used for simulations are based on values in Table 4.1 [11].

<table>
<thead>
<tr>
<th>Elevation (m)</th>
<th>Pressure (Torr)</th>
<th>Collision Rate (u) GHz</th>
<th>Electron Density ($n_e$) [$10^{14}$/m$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>27490</td>
<td>15</td>
<td>0.92</td>
<td>6.28</td>
</tr>
<tr>
<td>21554</td>
<td>35</td>
<td>1.46</td>
<td>2.69</td>
</tr>
</tbody>
</table>

The applied voltage was 10 kV (peak-to-peak) with a frequency of 5 kHz. By setting up these parameters, plasma with electron density of $n_e = 6.28 \times 10^{14}$/m$^3$ will be generated for 15 Torr, and $n_e = 2.69 \times 10^{14}$/m$^3$ for 35 Torr. In other words, the S11 difference between the two cases, when the conductive sheet is not covered by a plasma layer and when it is covered by plasma is simulated and measured. A reduction of 0.5 dB in reflected power from the conductive layer is produced for frequencies around 1.5 GHz revealed in Figure 4.2(a). Maximum RCS reduction is predicted to occur for frequencies close to the electron-neutral collision rate which is 1.46 GHz at the pressure of 35 Torr.
atmosphere which is confirmed in both the measured and simulated results shown and compatible with plasma physics predictions [12]. The differences between measured and simulated S11 curves are due in part to reflections in the experimental setup which are not present in the more ideal simulations. In Figure 4.2(b) the S11 difference for both cases is measured and simulated for X-band. As can be seen the reflected power reduction is less than that for the case with f=1.5 GHz, as predicted by theoretical plasma physics. By increasing the incident wave frequency the plasma becomes more transparent and so the attenuation decreases.

Figure 4.1 The experimental set up used for measuring the influence of plasma on RCS. (a) Top view. (b) Side view.

For this measurement, the power source is synchronized with the Vector Network Analyzer (VNA), so the reflection parameter (S11) is measured at maximum electron
density when the applied voltage is at peak. To do this, a visual basic code was written using predefined VNA objects enabling automatic measurement of specific frequency points within a specified frequency band.

Figure 4.2 S11 difference between a conductive plate with the plasma region present compared to the case of no plasma. The glass tube remains in place in both cases. Absorption by the plasma is small for both simulated and measured results showing good agreement over the frequency range. (a) S11 difference using WR770 (0.95 to 1.8 GHz). (b) S11 difference using X-band (8.2 to 12.4 GHz, WR90).

The RCS can be reduced theoretically using plasma installed at specific locations on a target but it is very difficult or even impossible to create such a scenario in a realistic setting. In particular at high atmospheric pressure, the electron density is sufficiently high
that collisions in the plasma occur at very high rates making the generation of plasma challenging and RCS reduction more difficult. We show through numerical modelling and experiment that there are measurable effects of plasma under high altitude conditions but these correspond to altitudes beyond conventional aircraft [11], for example, 15 Torr corresponds to an altitude of 27 km. It is possible to design theoretical plasma capable of providing RCS reduction. If one can arbitrarily select the plasma collision rate and electron density and profile as well as the thickness then it can be made to absorb radar signals. This has been shown in [12]-[15]. However, practical or even laboratory implementation of these scenarios seems difficult. The collision rate is determined by the atmospheric pressure and electron temperature and density; and at normal atmospheric pressure the recombination rates are too high for significant absorption to occur [16]. In addition to this is the need to generate plasma throughout a volume and preferably change the plasma properties with position. Potential generation methods might include high power microwaves, or high power lasers. In either case, the energy requirements will be large producing an even more troublesome situation than without using plasma. That is, instead of reducing the visibility of the target one is now turning it into a powerful microwave or infrared beacon which would be easily detected. In spite of the limited influence of plasma, the DBD actuator investigated in subsection 4.2.2, shows that it can appreciably affect RCS.

4.2.2 DBD Actuators

Plasma generated by DBD actuators has nearly the same characteristics as air from the point of view of the incoming radar signal, that is, the relative permittivity is close to one because of the high collision rate at atmospheric pressure [16]. This is further
constrained by the very spatially thin layer of plasma produced by the actuator, which is about 2 mm [6]. At lower pressures, corresponding to very high altitudes, the collision rates are low enough that the plasma will have electrical characteristics different than the surrounding air and there will be an opportunity for RCS reduction. Again the very thin nature of the actuator plasma could limit this effect.

Figure 4.3 shows the structure used for simulation and measurement. The dimensions of the copper and dielectric base are 30 cm×40 cm. The dielectric is then covered by 10 DBD actuators formed from copper strips with dimensions of 20 mm×300 mm separated by 15 mm×300 mm gaps. The copper tape has a thickness of 0.2 mm.

![DBD actuator structure](image)

**Figure 4.3 DBD actuator structure used for simulations and measurements.**

A plasma layer covers the gap between copper strips with a thickness of 2 mm. For air pressure of 15 Torr, an excitation voltage of 10 kV peak-to-peak at a frequency of 5 kHz, a plasma frequency and collision rate of $\omega_p = 1.42$ Grad/s and $\nu = 1$ GHz respectively are obtained. For the simulations, a commercial software (CST-MWS) is used.

Figure 4.4 shows the monostatic electric field of the DBD actuator at 10 GHz when the incident wave is vertically polarized or the E field is in +z direction. The angles $\theta$ and $\varphi$
are defined with respect to the z and x axis respectively. This simulation shows that the RCS increases for both field components ($E_\theta$ and $E_\phi$) in the presence of the plasma.

Figure 4.4 Monostatic Electric field of DBD actuator with 10 elements at 10 GHz for vertically polarized incident wave. (a) $E_\theta$ ($0 \leq \phi \leq 90$). (b) $E_\phi$.

Figure 4.5 shows the monostatic electric field of the same structure of Figure 4.3 for incident E-field polarized along the stripes or horizontal polarization. In other words, electric field of incident wave has non zero component in (+z) direction and $\phi$ varies from $0^\circ$ to $180^\circ$. These results show the increased RCS induced by the plasma for the actuator presented in Figure 4.3.

Figure 4.5 Monostatic Electric field of DBD actuator with 10 elements at 10 GHz for horizontally polarized wave. (a) $E_\theta$ (b) $E_\phi$. 
4.2.3 Bragg diffraction from DBD plasma actuator

Plasma actuators are used on aircraft for drag reduction or propulsion improvement. There is a significant body of literature describing and optimizing actuators for the highest performance as mentioned in Section 4.1. As we have shown at high pressure the actuators do not have any significant effect on RCS while at lower pressures the plasma layer can slightly lower RCS. An important factor which is ignored in the previous research is that for some angles actuators may even increase RCS due to Bragg diffraction.

![Figure 4.6 Diffraction from a periodic structure.](image)

Coherent illumination of an array of dielectric or conductive gratings will result in directional scattering of the incident wave. When electromagnetic waves are scattered from a crystal lattice or grating, peaks of scattered intensity are observed when the path length difference is equal to an integer number of wavelengths.

The condition for maximum intensity of bi-static RCS occurs for a specific diffraction angle as defined in (4-1).

\[
sin\theta_d - sin\theta_i = m \frac{\lambda/n}{\Lambda_g}
\]

(4-1)

Where \( \lambda \) is the wavelength of incident wave, \( \Lambda_g \) is the distance between two array elements, and \( \theta_i \) and \( \theta_d \) are the incidence and diffraction angles. For the plasma actuators
discussed in this thesis, the conductor strips act as crystal units. In this case, the grating diffraction reduces to the Bragg diffraction and (4-1) changes to (4-2) [17].

$$2A_g \sin \theta_d = m \frac{\lambda}{n}$$  \hspace{1cm} (4-2)

For the simulations, two different actuators were used. For the first one, the following parameters were used: ($A_g = 80$ mm, $w = 35$ mm, $f = 5.8$ GHz) and for the second one, the following parameters were used: ($A_g = 35$ mm, $w = 20$ mm, $f = 8$ GHz). Figure 4.7(a) shows the result of simulations for the backscattered wave in terms of the electric field. To find the Bragg diffraction angle, the simulation software Computer Simulation Technology-Micro Wave Studio (CST-MWS) was used. A plane wave with electric field in z direction is simulated using different parametric variables so it scans the incidence angles from $0^\circ$ to $90^\circ$ in one degree steps. Two different radar frequencies were used for these simulations (5 and 8 GHz). Figure 4.7(a) presents two peaks at 71 degrees (red curves) and 58 degrees (blue curve) which correspond to the scattering angle resulting from the Bragg diffraction phenomena. This corresponds to the diffraction angles $19^\circ$ and $32^\circ$ respectively. This is in good agreement with calculations using equation (4-2) which results in $\theta_d = 18.86^\circ$ and $\theta_d = 31.85^\circ$.

Figure 4.7(b) shows the measured S11 from a 6-element plasma actuator at 8GHz. The width of the copper strips is 2 cm and the gap between two strips is 1.5 cm. Three peaks can be observed on this figure. The biggest one is associated to the specular reflection which happens at $90^\circ$ and the two other peaks occur, one at $58^\circ$ and the other at $122^\circ$. These two peaks occur because of Bragg diffraction which for 8 GHz happens at 58 degrees (32 degrees from $90^\circ$) using (4-2), so there is a good agreement between theory and experiment.
Figure 4.7 (a) Mono-static reflection of wave from actuators showing the angles of maximum backscattered electric field. (b) The measured mono-static reflection from plasma actuator at 8 GHz.

### 4.3 Conclusion

Plasma actuators operating in the lower altitudes produce plasma with a very high collision rate and low electron density. Under these conditions, the dielectric properties of the plasma are not discernable from the air with our very sensitive experimental procedure. As a result, at high pressure, the actuator has no significant effect on RCS. In contrast, for higher altitudes the air pressure decreases so electron lifetime increases and the collision rate reduces, so the effect of plasma on incident electromagnetic waves can
be observed. Moreover, using conventional plasma actuators increases scattering at specific angles because of Bragg diffraction. This angle depends on the frequency and the spatial period of the actuator.
4.4 References


Chapter 5-Comprehensive RCS Simulation of Dispersive Media using SO-FDTD-DPW Method

5.1 Introduction

In this chapter the RCS from a plasma slab and the effect of different plasma profiles on RCS will be investigated. The simulations are performed using a split-field finite-difference time-domain (SF-FDTD) formulation with a perfectly matched layer (PML) boundary. These results show good agreement with plasma physics predictions. This code is written in MATLAB and runs on a PC Intel i5, taking 9 hours for a single case. SF-FDTD PML is a straightforward method to implement such simulations and enables us to control reflection on the PML-free space interface. Discrete plane wave (DPW) method is a good choice to generate plane wave since this method decreases TF/SF power leakage to -300 dB, and makes it suitable for small object RCS simulations. The SO-FDTD method has proven to be a stable and reliable method to model dispersive media and is thus suitable for modelling DBD actuators.

5.2 Background

One of the first problems in electromagnetic modelling is the formulation of absorbing boundaries to simulate infinite space and prevent reflections resulting from grid truncation. There are two popular absorbing boundary conditions (ABC) in use today, the Mur method [1], [2] and the perfectly matched layer (PML) method [3]. Until, 1993 all absorbing boundary conditions including Mur were limited in their ability to treat grazing incident waves. In 1994, Berenger proposed split-field PML (SF-PML) for 2D and in
1996 he developed a formulation to be employed for 3D space [4], [5]. The perfectly matched layer (PML) is generally considered the state-of-the-art for the termination of FDTD grids. There are some situations where specially designed ABC’s can outperform a PML, but this is very much the exception rather than the rule [6], [7]. The theory behind a PML is typically perfect in the continuous world, for all incident angles and for all frequencies. However, when a PML is implemented in the discretized world of FDTD, there are always some imperfections or reflections present.

Another challenging part in the modelling and simulation problems of electromagnetic waves is plane wave implementation. One of the most significant problems in electromagnetics is to predict the radar cross-section (RCS) of an object where it is assumed that the source is far away. For accurate calculation or prediction of RCS, one needs to have a good approximation of a plane wave. Plane waves have many other applications including telemetry or object-detection in multilayer structures. The first method proposed for numerical plane wave implementation was by Sadiku [8]. In 1995, Taflove proposed the total-field/scattered-field (TFSF) formulation outlined in [9]. This technique was perfectly developed for 1D, 2D and 3D spaces. In the finite-difference time-domain (FDTD) method, scattering problems are most often simulated by using the total-field/scattered-field (TFSF) formulation.

In order to implement TFSF, the computational domain is separated into two regions: the total field (TF) region where the scattering object, and both incident and scattered fields reside; and the scattered field (SF) region, where only the scattered fields, the output of interest, reside. There is a TFSF boundary which separates the grid into two regions TF and SF. There are two nodes adjacent to this boundary. One is in the SF
region and depends on a node in the TF region. The other is in the TF region and depends on a node in the SF region. To obtain self-consistent update equations, when updating nodes in the TF region, one must use the total field which pertains at the neighboring nodes. Conversely, when updating nodes in the SF region, one must use the scattered field which pertains at neighboring nodes.

An absorbing boundary condition simulates the open problem, and serves to cancel unwanted numerical reflections corrupting the scattered field. With the advent of PML, these reflections can be reduced to practically any desired level of accuracy. The TF/SF technique is based on a lookup table generated by interpolation between two adjacent cells on a 1D auxiliary grid. Introducing the lookup table decreases the memory needed for computation and makes this method computationally efficient. The plane wave sources are introduced at the Huygen’s surface to account for the discontinuity between the TF and SF regions. The power leakage achieved by the TF/SF method is -40 dB for the best condition regarding the wave incident angle. One part of this error comes from interpolation error and another part is a result of numerical dispersion mismatch between the 1D incident field array (IFA) and 3D wave propagation in TF region. To improve the IFA method, signal processing techniques [10] and optimizing the dispersion mismatch using matched numerical dispersion (MND) [11], [12] were proposed. Combining these two techniques reduces the power leakage caused by non-physical reflections on the Huygen’s surface to -70dB.

Another technique proposed to implement a numerical plane wave is Analytic Field Propagator (AFP) [13], [14]. In this method, the plane wave function is constructed in the frequency domain directly from the dispersion relationship. In the Yee FDTD grid, it is
possible to calculate the time series at an arbitrary point, given the incident field at a reference point and the direction of propagation. This allows one to realize an exact TFSF boundary for arbitrary directions of propagation. This method provides -180 dB of power isolation.

The method used in this work is the discrete plane wave (DPW) proposed by Tan in 2007 for 1-D space [15] and then developed for 3-D [16]. In contrast to IFA method, this method uses six 1-D auxiliary grids for each field and eliminates the need for interpolation. There is no dispersion mismatch and so it leads to a perfect plane wave implementation and TF/SF power leakage for this method is less than -300 dB. A 12-layer Split-Field PML is used for the ABC. The dispersive media is modelled by shift-operator FDTD [17]-[19]. Near-to-Far Field transformation (NTFT) is applied to calculate RCS in far-field [20]. This NTFT method is based on the surface equivalence theorem described by Balanis [21]. Finally the Fast Fourier Transform (FFT) is used to transform time domain signals to the frequency domain.

### 5.3 Formulation

#### 5.3.1 Split-Field PML

The split-field PML works by separating the electric and magnetic fields of the incident wave into two components on the ABC, the component whose plane of integration is tangential to the ABC surface, and the one normal to that. Only the latter needs to be attenuated by employing the ABC via the creation of an artificial biaxial anisotropic media that selectively apply the loss. These increase the number of Maxwell’s
equations from six to twelve equations. By using some simplifications the number of equations can be reduced to ten equations as they are described in (5-1) to (5-10).

\[
\mu_0 \frac{\partial H_{xy}}{\partial t} = \frac{\partial E_z}{\partial y}
\]  
(5-1)

\[
\mu_0 \frac{\partial H_{xz}}{\partial t} + \sigma_m H_{xz} = \frac{\partial}{\partial z} (E_{yx} + E_{yz})
\]  
(5-2)

\[
\mu_0 \frac{\partial H_{yz}}{\partial t} + \sigma_m H_{yz} = \frac{\partial}{\partial z} (E_{xy} + E_{xz})
\]  
(5-3)

\[
\mu_0 \frac{\partial H_{yx}}{\partial t} = \frac{\partial E_z}{\partial x}
\]  
(5-4)

\[
\mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial}{\partial x} (E_{yx} + E_{yz}) + \frac{\partial}{\partial y} (E_{xy} + E_{xz})
\]  
(5-5)

\[
\varepsilon_0 \frac{\partial E_{xy}}{\partial t} = \frac{\partial H_z}{\partial y}
\]  
(5-6)

\[
\varepsilon_0 \frac{\partial E_{yz}}{\partial t} + \sigma_z E_{xz} = \frac{\partial}{\partial z} (H_{yx} + H_{yz})
\]  
(5-7)

\[
\varepsilon_0 \frac{\partial E_{yx}}{\partial t} + \sigma_z E_{yz} = \frac{\partial}{\partial z} (H_{xy} + H_{xz})
\]  
(5-8)

\[
\varepsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial x}
\]  
(5-9)

\[
\varepsilon_0 \frac{\partial E}{\partial t} = \frac{\partial}{\partial x} (H_{yx} + H_{yz}) + \frac{\partial}{\partial y} (H_{xy} + H_{xz})
\]  
(5-10)

These equations can be discretized to be suitable for simulations. As long as the impedance of the lossy material matches the impedance of the interior domain which is free space then the reflection from the ABC interface can be eliminated. This matching condition is shown in (5-11).

\[
\frac{\sigma}{\sigma_m} = \frac{\varepsilon}{\mu}.
\]  
(5-11)

where \(\sigma\) and \(\sigma_m\) are electric and magnetic conductivities. In order to have no reflection on the PML interface, \(\sigma\) must be increased gradually from the interface surface outward. The back of the PML is usually terminated by a PEC layer. In theory, \(\sigma\) and \(\sigma_m\) are
continuous, but when field equations are discretized both \( \sigma \) and \( \sigma_m \) must be discretized. To optimize the value of these conductivities in different layers some research has been conducted [4], [22]. Within the PML, the conductivity was scaled using polynomial scaling [22].

\[
\sigma_\zeta(\zeta) = \frac{\sigma_{\zeta_{\text{max}}}|\zeta-\zeta_0|^m}{d^m}, \quad \zeta = x, y, z . \tag{5-12}
\]

Where \( \zeta_0 \) is the interface, \( d \) is the depth of the PML, and \( m \) is the order of the polynomial. A choice for \( \sigma_{\zeta_{\text{max}}} \) to minimize reflection is as [20]

\[
\sigma_{\zeta_{\text{max}}} = \sigma_{\text{opt}} \approx \frac{m+1}{150\pi\Delta\zeta} . \tag{5-13}
\]

### 5.3.2 Discrete Plane Wave

The Discrete plane wave method was proposed by Tengmeng in 2010 [16] to overcome errors generated in the IFA method from interpolation and numerical dispersion mismatch. In contrast to IFA this method employs six 1-D auxiliary grids where each grid is used to generate one of the electric or magnetic fields. In general, spacing for any grid can be non-uniform. It means \( \Delta r_x \neq \Delta r_y \neq \Delta r_z \), but such non-uniformity makes the relation among all six components very difficult to evaluate. However, the spacing can be uniform and the relation between six auxiliary grids can be evaluated for specific angles of incidence called rational angles. In this case, \( \Delta \zeta = p_\zeta \Delta \zeta = m_\zeta \Delta r \), where \( m_\zeta \) is an integer and \( p_\zeta \) is any of three components of propagation vector direction, \( P = (p_x, p_y, p_z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). The angles \( \theta \) and \( \phi \) are azimuthal and polar angles used for subspace projections. Using this definition any point in the main 3D grid can be found by the following equation:
\[ r \equiv i_r \Delta_r = p_x I \Delta_x + p_y J \Delta_y + p_z K \Delta_x = [l m_x + j m_y + k m_z] \Delta_r. \quad (5-14) \]

Where I, J and K are the number of cells at the same direction of x, y and z. Figure 5.1 clarifies this concept for the yz plane and it can be developed to yx and xz planes. The rational angles for which (5-14) comes true can be calculated as:

\[
\begin{aligned}
\sin \theta &= m_y \Delta_x / \sqrt{m_x^2 \Delta_x^2 + m_y^2 \Delta_y^2} \\
\tan \theta &= \sqrt{p_x^2/p_y^2} \\
p_{\zeta} &= \frac{m_{\zeta} \Delta_r}{\Delta \zeta}.
\end{aligned}
\quad (5-15)
\]

Figure 5.1 Reconstruction of fields at any point of main grid based on auxiliary grid fields.

Since there are infinite combinations on \( m_{\zeta} \) and \( \Delta \zeta \), there would be an infinite number of incident angles at which the plane wave can be generated in the main grid.

The electric fields and magnetic fields must be constructed using (5-16) and (5-17), which are discretized Maxwell’s equations.

\[
H_{\zeta+1} \bigr|_{i_r + m_{\zeta} + m_{\zeta+1}/2} = C_{ha} \cdot H_{\zeta} \bigr|_{i_r + m_{\zeta}/2} + C_{hb,\zeta+1} \cdot E_{\zeta+2} \bigr|_{i_r + m_{\zeta} - m_{\zeta+1}/2} -
\]

\[
E_{\zeta+2} \bigr|_{i_r + m_{\zeta} + m_{\zeta+1}/2} + C_{hb,\zeta+2} \cdot E_{\zeta+1} \bigr|_{i_r + m_{\zeta} + m_{\zeta+2}/2} -
\]

\[
E_{\zeta+1} \bigr|_{i_r + m_{\zeta} - m_{\zeta+1}/2}.
\quad (5-16)
\]
\[ E_\zeta \left|_{i_r + m_\zeta + 1 + m_\zeta + 2/2} \right. = C_{ea} \cdot E_\zeta \left|_{i_r + m_\zeta + 1 + m_\zeta + 2/2} \right. + C_{eb,1+1} \left[H_{\zeta + 2} \left|_{i_r + m_\zeta + 1 + m_\zeta + 2/2} \right. - H_{\zeta + 1} \left|_{i_r + m_\zeta + 1 + m_\zeta + 2/2} \right. \right] + C_{eb,2+1} \left[H_{\zeta + 1} \left|_{i_r + m_\zeta + 1/2} \right. - H_{\zeta + 1} \left|_{i_r + m_\zeta + 1 + m_\zeta + 2/2} \right. \right], \quad (5-17) \]

where \( \zeta = x, \zeta + 1 = y \) and \( \zeta + 2 = z \). In these equations \( C_{ha}, C_{hb}, C_{ea} \) and \( C_{eb} \) are the constants which are related to the time step and the cell size of the grid.

Another significant aspect of the DPW method is source generation. In this method hard sources have to be used instead of soft sources. If \( \psi \) is the polarization angle defined in [9] then E-field polarization projection can be defined as \( P'_{\zeta, \psi} \). For instance, \( P'_{x, \psi} = \sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi \). If \( f_H(t_n) \) is the source function applied to the grid at reference point \( X_0 = (0,0,0) \), then for any spatial shift \([m_\zeta/2 + i_r] \Delta_r \) a time delay \([m_\zeta/2 + i_r] \Delta_r \sqrt{\varepsilon \mu} \) will occur. A plane wave in the 3-D grid can be generated by E-field using (5-16) and (5-17) where the hard source is imposed based on (5-18).

\[ E_\zeta \left|_{i_r + m_\zeta /2} \right. = P'_{\zeta, \psi} \cdot f_E(t_{n+1} - \left[\frac{m_\zeta}{2} + i_r \right] \Delta_r \sqrt{\varepsilon \mu}), \quad \text{for} \quad 0 \leq i_r \leq \max \{\lfloor m_\zeta \rfloor \}. \quad (5-18) \]

Although, one can generate a plane wave using (5-16), (5-17) and (5-18), the shape of the plane wave generated by this method is not exactly the same as the initial hard source function. The reason for that is the numerical reflection on 1-D grids due to the truncation. In order to cope with this issue we use the Optimized Analytic Field Propagator (O-AFP) [13], [14]. In this method the Fourier transform of the initial wave function must be calculated and then since the wave number \( k(\omega) \) is frequency dependent the delay at any point of 1-D grid can be calculated for each frequency by introducing a phase factor as shown in (5-19)
\[ E_{inc,\xi} |_{r_{\xi}} = E_{inc} |_{r_{ref}}, P'_{\xi,\psi}, e^{-jk(\omega)(r_{\xi} - r_{ref})}, \]  

where \( r_{ref} \) is the reference point on the 1-D grid where the hard source is applied. Having the spectrum of the signal at any point, the time domain signal at all points on the 1-D grid can be determined by using inverse Fourier transform.

### 5.3.3 Shift-Operator FDTD

The technique used to simulate plasma as a dispersive media is SO-FDTD. This method is based on the Auxiliary Differential Equation (ADE) method in which the permittivity of the dispersive medium is expanded as a rational fraction function as

\[ \varepsilon_r(\omega) = \sum_{n=0}^{N} p_n(j\omega)^n \sum_{n=0}^{N} q_n(j\omega)^n. \]  

(5-20)

Then by using \( j\omega \rightarrow \partial / \partial t \), the frequency domain can be transformed to time-domain. This way the third differential equation can be made from the frequency domain equation

\[ \mathbf{D}(t) = \varepsilon_0 \varepsilon_r \left( \frac{\partial}{\partial t} \right) \mathbf{E}(t). \]  

(5-21)

If central difference is used, (5-21) can be discretized. Assume \( y(t) = \partial f(t) / \partial t \), then one can write

\[ \frac{y^{n+1} + y^n}{2} = \frac{f^{n+1} - f^n}{\Delta t}. \]  

(5-22)

In general, \( z_t f^n = f^{n+1} \), where \( z_t \) is the shift operator. The (5-22) can be written as:

\[ y^n = \left( \frac{2}{z_t - 1} \right) f^n. \]  

(5-23)

\[ \partial / \partial t \rightarrow \left( \frac{2}{\Delta t z_t + 1} \right). \]  

(5-24)

Substituting (5-24) and (5-20) in (5-21), the constitutive relationship for dispersive media in the time domain can be written as
\[
\left[ \sum_{k=0}^{K} q_k \left( \frac{2 z t - 1}{\Delta t z t + 1} \right)^k \right] D^n = \varepsilon_0 \left[ \sum_{k=0}^{K} p_k \left( \frac{2 z t - 1}{\Delta t z t + 1} \right)^k \right] E^n. \quad (5-25)
\]

Considering the non-magnetized plasma permittivity as
\[
\varepsilon_r = 1 + \frac{\omega_p^2}{j\omega(j\omega+\nu)}, \quad (5-26)
\]
where \(\omega_p\) and \(\nu\) are the plasma frequency and collision rate. We can identify the coefficients \(q_k\) and \(p_k\); so \(p_2 = 1, p_1 = \nu, p_0 = \omega_p^2\) and \(q_2 = 1, q_1 = \nu, q_0 = 0\). Having these coefficients, the update equation for electric fields resulting from (5-25) would be:
\[
E^{n+1} = \frac{1}{\varepsilon_0(\omega_p^2\Delta t^2 + 2\nu\Delta t + 4)} [(2\nu\Delta t + 4)D^{n+1} - 8D^n + (-2\nu\Delta t + 4)D^{n-1} -
\varepsilon_0(2\omega_p^2\Delta t^2 - 8)E^n - \varepsilon_0(\omega_p^2\Delta t^2 - 2\nu\Delta t + 4)E^{n-1}] . \quad (5-27)
\]

### 5.3.4 Near-to-Far Field Transformation (NTFT)

In the FDTD method, the fields are calculated in the near-field, so they must be transformed to the far-field to be able to calculate RCS. This NTFT transformation is based on the equivalence surface theorem described in [19] and [20]. Two radiation vectors \(\mathbf{N}\) and \(\mathbf{L}\) are calculated using two integral equations
\[
\mathbf{N} = \int_{S'} \mathbf{J}_s \exp(jkr'.\hat{r}) dS', \quad (5-28)
\]
\[
\mathbf{L} = \int_{S'} \mathbf{M}_s \exp(jkr'.\hat{r}) dS', \quad (5-29)
\]
where harmonic scattered surface currents \(\mathbf{J}_s = \hat{n} \times \mathbf{H}\), and \(\mathbf{M}_s = \mathbf{E} \times \hat{n}\) exist on the surface, when \(H\) and \(E\) are the scattered magnetic and electric fields at the surface. Also, in (5-28) and (5-29) \(k\) is the wavenumber, \(\hat{r}\) is the unit vector to the far zone field point, \(r'\) is the vector to the source point of integration, and \(S'\) the closed surface surrounding
the scatterer. After finding \( N \) and \( L \), the time harmonic far zone scattered electric fields are

\[
E_\theta = -j \exp(jkR)(\eta N_\theta + L_\theta)/(2\lambda R),
\]

\[
E_\phi = -j \exp(jkR)(-\eta N_\phi + L_\phi)/(2\lambda R),
\]

(5-30)

(5-31)

where \( \eta \) is the impedance of free space and \( R \) is the distance from origin to the far zone field point. By discretizing these two equations, they can be used for numerical simulations. However for some specific conditions like strongly forward scattering objects the integral is not needed to be calculated on the whole surface [23].

5.4 Simulation Results

Based on the theory explained in previous sections, simulation is performed on dispersive media with permittivity specified in (5-26). For this simulation a Gaussian derivative pulse as shown in (5-32) and Figure 5.2 is used to make the plane wave

\[
E_x(t) = (10^{-5}) \left( \frac{2(t-3T)}{T^2} \right) \exp\left(\frac{-((t-3T)/T)^2}{\tau^2}\right). \tag{5-32}
\]

Figure 5.2 The input wave function to generate plane wave.
The simulation space is a $70 \times 70 \times 70$-cell cubic region. Twelve layers from each side are used for SF-PML as the absorbing boundary and the conductivity in this anisotropic region is chosen to have a reflection coefficient of $10^{-7}$ on the PML-Free space interface. There are 3-cell layers between the PML and SF/TF interfaces. This region is the scattered field region from which the far fields are calculated. The TF region is a $40 \times 40 \times 40$-cell cubic region and the scatterer is located in this region as shown in Figure 5.3.

Figure 5.3 A YZ cut plane in the middle of SF/TF region when scatterer is located in TF region.

For the simulation we choose $T=0.1\text{ns}$, $\Delta t = 5\text{ps}$ and space step ($\Delta = 3\text{mm}$). If we select $T=0.1$ ns it means that the bandwidth of the signal is $\text{BW}=2/T$ or 20 GHz. In order to meet Courant and Nyquist conditions for stability and sampling accuracy $\Delta t$ and $\Delta$ are selected such as $\Delta t = T/20$. In order to prove that the SF-PML layer works, a dipole antenna is located in the middle of the TF region and then the response is compared with the analytical solution of the dipole antenna at 0.5 m from the dipole antenna. As shown in Figure 5.4 there is good agreement between simulation and analytic solutions. Figure
5.4(a) shows the signal in the time-domain and Figure 5.4(b) is the frequency response using FFT. To further verify the simulator, a parallel polarized wave is applied when there is no object in the TF region and then power leakage on TF/SF interface is calculated as shown in Figure 5.5. The formula used to calculate this graph is $20 \log_{10} \left( \frac{E_x}{E_z} \right)$ which gives a number equal to 300 dB in terms of power leakage isolation between TF/SF interface.

Figure 5.4 Comparison between simulation result and analytic solution of a short dipole antenna at 0.5 m from antenna.

Figure 5.5 Electric field intensity isolation on TF/SF interface for DPW method.
To verify the fields in the TF region, a dielectric sphere is located in the TF region and the normalized electric fields ($E_y$ and $E_z$) are calculated by applying a sine wave at $f_0 = 2.5 \text{ GHz}$. These fields are compared with the Mie exact solution [24] for dielectric permittivity of ($\varepsilon_r = 4$). Figure 5.6 shows these fields on the axis of the dielectric sphere. The sphere radius is 15 cells (4.5 cm) and the center is at cell number 35.

![Normalized $E_y$ for points near dielectric sphere axis](image1)

![Normalized $E_z$ for points near dielectric sphere axis](image2)

Figure 5.6 Normalized electric fields ($E_y$ and $E_z$) on the axis of dielectric sphere. a) $\frac{E_y}{E_0}$, b) $\frac{E_z}{E_0}$.

![Normalized RCS for PEC sheet](image3)

Figure 5.7 Comparison between measured and simulated RCS of PEC at different wavelengths.
In order to verify the RCS achieved by the designed code a thin PEC sheet with the thickness of \(d\) and the side edge lengths of \(a\) and \(b\) (\(d=3\) mm, \(a=93\) mm and \(b=93\) mm) was located in the TF region and the RCS was compared to measured RCS for a few specific frequencies [25]. Figure 5.7 shows the calculated and measured RCS of this PEC sheet.

Figure 5.8 reveals the RCS of a plasma slab with a thickness of one cell for different \(\omega_p\) when \(v = 1GHz\). Figure 5.9 shows the RCS of the same plasma slab for various collision rates when \(f_p = 5GHz\). Both figures are in agreement with plasma physics predictions. Figure 5.10 compares the RCS of a Perfect Electric Conductor (PEC) when it is not covered by a plasma layer and when it is covered by a plasma layer of different thickness. For this simulation the plasma frequency and collision rate are (\(f_p = 5GHz, v = 15GHz\)).

![RCS of plasma slab with thickness of 3 mm for different plasma frequencies (fp)](image)

Figure 5.8 RCS of plasma slab in respect to different plasma frequencies (\(\omega_p\)).
Figure 5.9 RCS of plasma slab of 3 mm thickness for various collision rates.

Figure 5.10 RCS of Perfect Electric Conductor (PEC) when it is not covered by a plasma layer and when it is covered by a plasma layer of various thicknesses.

Figure 5.11 shows the RCS of a PEC sheet covered by a 6 mm plasma layer for different incident angles. Since the object is symmetrical the angles are reported from 0 to 90 degree. The angle ($\phi$) is in respect to the +x axis, so the maximum specular reflection occurs for $\phi = 90^\circ$. 
Figure 5.11 RCS of PEC (a=93 mm and b=93 mm) covered by 6 mm plasma layer ($E = E_z$).

All previous simulations are done for the electric field in the +z direction or parallel to the sheet. Figure 5.12 shows the monostatic RCS of the same PEC sheet when the electric field is in the +x direction, so the angle between incident electric field and the surface of the sheet will be changed at each step.

A PEC sheet is covered by two different plasma profiles and the FDTD method is used to simulate the specular reflection coefficient (R) from this layer. Equation (5-33) shows the complex refractive index for the film covering the conductive sheet. The equations for these two profiles are shown in (5-34) and (5-35).

$$\hat{n}_f(y) = n_f(y) + jk_f(y),$$  \hspace{1cm} (5-33)

$$n_f(y) = n_{f0} + (n_{fd} - n_{f0})\left((d_f - y)/d_f\right)^a, \text{ and}$$

$$k_f(y) = k_{f0} + (k_{fd} - k_{f0})\left((d_f - y)/d_f\right)^a, \hspace{1cm} (5-34)$$
\begin{align*}
  n_f(y) &= B_n + A_n e^{(a(d_f-y)/d_f)} \quad \text{and} \quad k_f(y) = B_k + A_k e^{(a(d_f-y)/d_f)}, \\
  \text{where} \quad A_n &= \frac{(n_{fd}-n_{f0})}{e^{a-1}}, \quad B_n = -\frac{(n_{fd}-n_{f0}e^a)}{e^{a-1}}, \quad A_k = \frac{(k_{fd}-k_{f0})}{e^{a-1}} \quad \text{and} \quad B_k = -\frac{(k_{fd}-k_{f0}e^a)}{e^{a-1}}.
\end{align*}

While \( n_{f0} \) and \( k_{f0} \) are the real and imaginary parts of the refractive index of the film at the film-free space interface, \( n_{fd} \) and \( k_{fd} \) are the real and imaginary parts of the refractive index of the film at the film-substrate interface and \( a \) is the order of polynomial or exponential function. Figure 5.13(a) and (b) show reflection coefficients for different orders \( a \). The reflection coefficient is more sensitive to the order of the function or the gradient of the refractive index than the thickness of plasma layer or the angle of incidence. Figure 5.14(a) and (b) show the reflection coefficient for different film thickness \( (d_f) \) and Figure 5.15(a) and (b) show the reflection coefficient for different incident angles \( (\phi) \). It can be seen from these results that by choosing correct order of functions the minimum reflection is achieved. If the thickness of the plasma layer...
increases, the order of functions must increase to have minimum reflections. Figure 5.13(a-5) shows the reflection coefficient when the conductive sheet is covered by homogeneous and real permittivity dielectric. It can be seen that for different orders the reflection coefficient remains very close to one, but when the imaginary part is added to the refractive index the reflection decreases (Figure 5.13(a-6)). For Figure 5.14 simulations are performed for three different orders ($n = 3, 5$ and $7$) and it can be seen for both TE and TM cases the reflection decreases by increasing the thickness. Comparing Figures 13, 14 and 15, it can be revealed that reflection coefficient is more sensitive to the order of function or the gradient of refractive index than the thickness of the plasma layer or the angle of incidence.

Figure 5.13 Reflection coefficient from a conductive sheet covered by an inhomogeneous plasma profile (a) TE mode (solid) polynomial and (dotted) exponential. For graphs (1, 2 and 3) $k_{fd} = n_{fd} = 1, k_{f0} = 0, n_{f0} = 1$ (1) $d_f = 0.06 \, m$, (2) $d_f = 0.09 \, m$, (3) $d_f = 0.21 \, m$ (4) $k_{fd} = n_{fd} = 10^5, k_{f0} = 0, n_{f0} = 1, d_f = 0.21 \, m$ (5) $k_{fd} = 0, n_{fd} = 5, k_{f0} = 0, n_{f0} = 1, d_f = 0.21 \, m$ (6) $k_{fd} = 10, n_{fd} = 5, k_{f0} = 1, n_{f0} = 0, d_f = 0.06 \, m$ (7) $k_{fd} = n_{fd} = 10^5, k_{f0} = 0, n_{f0} = 1, d_f = 0.21 \, m$ (b) TM mode (dash-dash) polynomial and (dash-point) exponential (8 and 9) $k_{fd} = n_{fd} = 10^5, k_{f0} = 0, n_{f0} = 1, d_f = 0.09 \, m$ and $0.21 \, m$ respectively (10) $k_{fd} = n_{fd} = 1, k_{f0} = 0, n_{f0} = 1, d_f = 0.06 \, m$ (11 and 12) $k_{fd} = n_{fd} = 10^5, k_{f0} = 0, n_{f0} = 1, d_f = 0.06 \, m$ and $0.21 \, m$ respectively.
Figure 5.14 Reflection coefficient from a conductive sheet covered by an inhomogeneous profile (a) TE mode (solid) polynomial and (dotted) exponential. For graphs (1 and 2) $k_{fd} = n_{fd} = 1$, $k_{f0} = 0$, $n_{f0} = 1$ (1) $a = 3.5$ (2) $a = 5$ (3) $k_{fd} = n_{fd} = 10^5$, $k_{f0} = 0$, $n_{f0} = 1$, $a = 5$ (4) $k_{fd} = n_{fd} = 1$, $k_{f0} = 0$, $n_{f0} = 1$, $a = 5$ (5) $k_{fd} = n_{fd} = 10^5$, $k_{f0} = 1$, $n_{f0} = 0$, $a = 17$ (b) TM mode (dash-dash) polynomial and (dash-point) exponential (6) $k_{fd} = n_{fd} = 1$, $k_{f0} = 0$, $n_{f0} = 1$, $a = 3.5$ (7) $k_{fd} = n_{fd} = 1$, $k_{f0} = 0$, $n_{f0} = 1$, $a = 5$ (8) $k_{fd} = n_{fd} = 10^5$, $k_{f0} = 0$, $n_{f0} = 1$, $a = 5$ (9) $k_{fd} = n_{fd} = 1$, $k_{f0} = 0$, $n_{f0} = 1$, $a = 5$ (10) $k_{fd} = n_{fd} = 1$, $k_{f0} = 0$, $n_{f0} = 1$, $a = 17$

Figure 5.15 Reflection coefficient from a conductive sheet covered by an inhomogeneous profile (a) TE mode (solid) polynomial and (dotted) exponential and (b) TM mode (dash-dash) polynomial and (dash-point) exponential. For all graphs $k_{fd} = n_{fd} = 1$, $k_{f0} = 0$, $n_{f0} = 1$, $d_f = 0.09 m$, (1, 3, 5 and 7) $a = 3.5$ (2, 4, 6 and 8) $a = 5$. 
5.5 CONCLUSION

Based on the simulation results, it can be seen that by adding the plasma slab, reduction in RCS happens for a specific range of spectrum. Note that this reduction caused by absorption in the plasma layer is very low and if the thickness of the plasma layer increases compared to the dimensions of the PEC, it causes an increase in RCS. For the layer with a thickness of 3 mm this reduction is about 0.2 $dBm^2$. Figures 5.11 and 5.12 reveal that by changing the polarization of the electric field, the reflection or RCS would be changed. The results of simulation of different profiles show that the reflected power from a conductive layer can be reduced if some specific profile is used. It is shown that the reflection cannot be changed just by using a material with real permittivity or homogeneous refractive index, but it decreases to a large extent by choosing inhomogeneous plasma for which the imaginary part of the permittivity is not equal to zero.
5.6 References


[23] X. Li, A. Taflove, and V. Backman, “Modified FDTD Near-to-Far-Field Transformation for Improved Backscattering Calculation of Strongly forward-


Chapter 6- Contributions, Conclusion, and Future Work

6.1 Contributions

The work of this thesis has encompassed simulation and measurement which has led to the correction of cold plasma parameters including electron density, plasma frequency and electron-neutral collision rate. A new method of measuring plasma properties based on the microwave cavity perturbation method is developed along with using the automated control of a vector network analyser. The interaction of electromagnetic waves with a DBD at atmospheric pressure was investigated and yielded previously unreported scattering due to Bragg diffraction. A comprehensive analysis and simulation of dispersive media radar cross section was done using the finite-difference time-domain method. The results are published in refereed journals and conference proceedings.

6.1.1 Cavity and Choke Design

The rectangular cavity and the glass tube enclosing the plasma region were designed. For the energy coupling to the cavity a rectangular waveguide was used and the rectangular aperture with curved ends was used to couple energy from the waveguide to the cavity. Chokes were designed as stepped impedances to prevent energy leakage from the glass tube holes on the broad side walls of the cavity.

6.1.2 Cold Plasma Parameters

A cylindrical DBD plasma under five different pressures was generated in an evacuated glass tube. This plasma volume was located at the center of a rectangular copper waveguide cavity where the electric field is maximum for the first mode and the magnetic field is very close to zero. The microwave perturbation method was used to
measure electron density and plasma frequency for these five pressures. Simulations by a commercial microwave simulator are comparable to the experimental results.

6.1.3 Interaction of Electromagnetic Wave with DBD Actuator

Plasma actuators operating in the lower altitudes produce plasma with a very high collision rate and low electron density. It was shown that under these conditions, the dielectric properties of the plasma are not discernable from the air with our very sensitive experimental procedure. As a result, at high pressure, the actuator has no significant effect on RCS. In contrast, for higher altitudes the air pressure decreases so electron lifetime increases and the collision rate is reduced, so the effect of plasma on incident electromagnetic waves can be observed. Moreover, it was simulated and measured that conventional plasma actuators cause increased scatterings at specific incident angles because of Bragg diffraction. These angles depend on the frequency of the incident wave and the spatial period of the actuator.

6.1.4 Comprehensive RCS Simulation of Dispersive Media Using using SO-FDTD-DPW Method

Simulations were conducted using SF-FDTD for 12-layer PML, and the plane wave was generated using the DPW technique and AFP. These results show good agreement with plasma physics predictions. It can be seen by increasing the collision rate that the RCS decreases and by increasing the plasma frequency, the RCS increases where the limit is the RCS of the PEC. For a thin layer of plasma the RCS decreases for a specific frequency band which is close to the collision rate. This phenomenon can be explained as absorption of the electromagnetic energy by the electrons in the plasma layer. This code is written in MATLAB and runs on a PC Intel i5, taking 9 hours for a single case. SF-
FDTD PML is a straightforward method to implement such simulations and enables us to control reflection on the PML-free space interface. The DPW is a good choice since this method decreases TF/SF power leakage to 300 dB, and makes it suitable for small object RCS simulations. The SO-FDTD method is proven to be a stable and reliable method to model dispersive media.

Based on the simulation results, it can be seen that by adding the plasma slab, reduction in RCS happens for a specific range of spectrum. Note that this reduction caused by absorption in the plasma layer is very low and if the thickness of the plasma layer increases compared to the dimensions of the PEC, it causes an increase in RCS. For the layer with thickness of 3 mm this reduction is about 0.2 dBm². Simulated results also reveal that by changing the polarization of electric field, the reflection or RCS would be changed. The results of simulation of different profiles show the reflected power from a conductive layer can be reduced if some specific profile is used. It is shown that the reflection cannot be changed just by using a material with real permittivity or homogeneous refractive index, but it decreases to a large extent by choosing inhomogeneous plasma for which the imaginary part of the permittivity is not equal to zero.

6.2 Conclusion

This thesis has presented a detailed simulation and experimental study of cold plasma characteristics at different pressures using the cavity perturbation technique. The plasma parameters have been measured for these specific pressures and the experimental result are comparable with simulations. This aids in the development of a fundamental and intuitive understanding of the behavior of cold plasma at different altitudes and radar
cross section investigations of the plasma layer. The DBD plasma actuator interaction with electromagnetic waves was measured and simulated at atmospheric pressure and it was shown that the geometry of the actuator can cause some extra scattering at specific angles.

In order to have a more trustworthy simulation result a computer code was developed to model dispersive media radar cross section and later this code was extended to simulate the RCS of anisotropic media and the effect of the plasma profile on radar cross section. The conclusion achieved from the simulations and measurements permitted us to understand how plasma can be used to decrease radar cross section, however implementing some of the results like having a specific plasma profile, is challenging.

6.3 Future Work

The result of experiments in cold plasma parameter measurement can be improved if a higher quality factor cavity can be designed. The results of simulation can be more accurate for other object geometries if the FDTD method is modified, and it can be faster if parallel or distributed computing is employed.

6.3.1 High Quality Factor Cavity

The accuracy of plasma parameters measured by the cavity perturbation method depends on the quality factor of the cavity. As a higher quality factor is achieved, the results would be closer to the actual values.

6.3.2 High Performance Computing

The simulation result accuracy is closely related to the cell size of the main grid. When the cell size decreases the simulation error decreases. The disadvantage is that
increasing the number of cells in the main grid increases the computational burden, and the used computer resources. By employing parallel processing or distributed computing the time spent for performing the simulation will be reduced.

### 6.3.3 Modified Numerical Method

The method used for this research is the FDTD method. However this method is straightforward and fully developed for electromagnetics problems, but the disadvantage is the cubic form of the cells. It causes the result for curved edges not to be accurate. Conformal mapping can be combined with FDTD for curved objects. Moreover, the finite element method can be employed for improved accuracy for different object geometries. However, the finite element method increases the memory needed for simulation. For large size objects one can use the parabolic wave equation to investigate the RCS.
Appendix A
VBScript Code for VNA Automation

'This program creates a S24 measurement, with 2.5 MHz bandwidth
'of waveguide cavity in S-band to measure peak in time domain
'for finding permittivity when plasma is on
'It is written in VBscript using COM commands

Dim PNA
'set objexcel=createobject("Excel.sheet")
set textbox=createobject("textbox.text")
set PNA = CreateObject("AgilentPNA835x.Application")
PNA.Preset

Set chan=PNA.activechannel
Set meas=PNA.activemeasurement
Set trce = PNA.ActiveNAWindow.ActiveTrace
meas.MarkerState(1)=True
PNA.activemeasurement.Marker(1).Format=nalinmag
Set Mark= meas.Marker(1)

meas.ChangeParameter "S24",1
PNA.TriggerSetup.Source=2

'chan.StartFrequency = 45e6
'chan.StopFrequency = 500e6
trce.ReferencePosition = 2
trce.ReferenceValue = -37.5
trce.YScale = 0.35
'PNA.TriggerSignal = 3
chan.TestPortPower(4) = 30
chan.NumberOfPoints = 200
chan.SweepType = 3
chan.IFBandwidth = 600e+3
'chan.sweepTime = 1.77e-3
chan.Averaging = True
chan.AveragingFactor = 50
swptme = chan.SweepTime
'chan.ExternalTriggerDelay = 10

public n
n = Cint(n)
n = 200
fs = (2.491e6)/n
const FsoForWriting = 8

'Do Test
dim Fso, f
dim dt(200)
'dim yvalue(10)

for i = 1 to n
f=2.834178e9+fs*i
chan.CWFrequency=f
chan.single true
'chan.ExternalTriggerDelay=150
PNA.TriggerDelay=0.1

yvalue=0
for j=1 to 12
chan.single true
meas.Activemarker.searchMax
yvalue=mark.value(1)+yvalue
next

yvalue=yvalue/12
'meas.Activemarker.searchMax
'yvalue=mark.value(1)
dt(i)=yvalue
meas.showstatistics=true
'msgbox("MaxPoint= " & dt(i))
'picbox.print yvalue
next

call SaveDataToFile
f.close()
set f=Nothing
sub SaveDataToFile
    set Fso=createObject("Scripting.FileSystemObject")

    set f=Fso.OpenTextFile("c:\Documents and Settings\Administrator\Desktop\testVB.txt",FsoForWriting,true)
    for i=1 to n
        f.writeline(dt(i))
    next
end sub

msgbox("Done Testing")
Appendix B

MATLAB® Code for Dispersive Media RCS Simulation Using Split-Field FDTD_DPW Method

% code based on 3D WITH SPLIT-FIELD PML
%(dispersive material based on so-FDTD-modified based on code8)
% SO-FDTD Method

clear all
ep0=8.854188*10^(-12); mio0=4*pi*10^(-7);
etha0=sqrt(mio0/ep0);
d=40*10^(-2); % thickness of plasma slab
d0=20*10^(-2);
th=5*10^(-2); % thickness of conductive layer
h=10*10^(-2); % height of conductive and plasma layer
w=20*10^(-2); % width of slabs
c=299792458;
zigmacopper=8*10^7;
f=1000*10^7;
omega=2*pi*f;
T=5*10^(-10);
P=12;

delta=(c/f)*(1/10)*(1/1.5);
deltat=deltat/(4*c);
Re=deltat/(2*ep0); Rh=deltat/(2*mio0); Ra=(deltat/(ep0*delta));
Rb=deltat/(mio0*delta);
M=48; N=48; P=48;
M=M+1, N=N+1, P=P+1,

ep=ones(M,N,P);
mio=ones(M,N,P);
zigma=zeros(M,N,P);
zigmam=zeros(M,N,P);

zigmax=zeros(M-1,N-1,P-1);
zigmay=zeros(M-1,N-1,P-1);
zigmaz=zeros(M-1,N-1,P-1);
zigmacof=1000000;
zigmawx=zeros(M,N,P);
zigmayw=zeros(M,N,P);
zigmawz=zeros(M,N,P);

Hx=zeros(M,N,P,3);
Hy=zeros(M,N,P,3);
Hz=zeros(M,N,P,3);
Ex=zeros(M,N,P,3);
Ey=zeros(M,N,P,3);
Ez=zeros(M,N,P,3);

Hxy=zeros(M,N,P,3);
Hyx=zeros(M,N,P,3);
Hxz=zeros(M,N,P,3);
Exy=zeros(M,N,P,3);
Eyx=zeros(M,N,P,3);
Ezx=zeros(M,N,P,3);
Hxz=zeros(M,N,P,3);
Hyz=zeros(M,N,P,3);
Hzy=zeros(M,N,P,3);
Exz=zeros(M,N,P,3);
Eyz=zeros(M,N,P,3);
Ezy=zeros(M,N,P,3);
Ca=zeros(M,N,P);
Cb=zeros(M,N,P);

%% Dispersive constants %%
Sc=c*deltat/delta;
g=3*10^9; % damping factor or collision rate
Ng=1/(g*deltat);
wp=20*10^9; fp=wp/(2*pi); lambdap=c/fp;
Np=lambdap/delta;

Cjj=(1-1/(2*Ng))/(1+1/(2*Ng));
Cje=(1/(1+1/(2*Ng)))*((2*pi^2*Sc)/(etha0*Np^2));
epinf=1;
i0D=19; j0D=23; k0D=19;
i1D=31; j1D=27; k1D=31;
ReD=deltat/(2*epinf*ep0); RaD=(deltat/(ep0*epinf*deltat));
ReD2=(Cje*etha0*Sc)/(2*epinf); RaD2=2*ReD2/Cje;
Dx=zeros(M,N,P,3);
Dy=zeros(M,N,P,3);
Dz=zeros(M,N,P,3);

%%%%% NTFT Parameters
R1=1.2; cvector=[25 25 25]*delta; %[29 29 29]*delta;
phip=2*pi-pi/2; thetap=pi/2; % The point we measure RCS
RU=[sin(thetap)*cos(phip) sin(thetap)*sin(phip) cos(thetap)];
R=R1*RU-cvector;
nnmax=fix(sqrt(R*R')/(c*deltat))+1;
U1x=zeros(nnmax); U1y=zeros(nnmax); U1z=zeros(nnmax);
U2x=zeros(nnmax); U2y=zeros(nnmax); U2z=zeros(nnmax);
U3x=zeros(nnmax); U3y=zeros(nnmax); U3z=zeros(nnmax);
U4x=zeros(nnmax); U4y=zeros(nnmax); U4z=zeros(nnmax);
U5x=zeros(nnmax); U5y=zeros(nnmax); U5z=zeros(nnmax);
U6x=zeros(nnmax); U6y=zeros(nnmax); U6z=zeros(nnmax);
W1x=zeros(nnmax); W1y=zeros(nnmax); W1z=zeros(nnmax);
W2x=zeros(nnmax); W2y=zeros(nnmax); W2z=zeros(nnmax);
W3x=zeros(nnmax); W3y=zeros(nnmax); W3z=zeros(nnmax);
W4x=zeros(nnmax); W4y=zeros(nnmax); W4z=zeros(nnmax);
W5x=zeros(nnmax); W5y=zeros(nnmax); W5z=zeros(nnmax);
W6x=zeros(nnmax); W6y=zeros(nnmax); W6z=zeros(nnmax);
eEtheta=0; eEphi=0; eEinc=0;

R0=10^(-7);
order=4; %
zigmaxmax=-(order+1)*ep0*c*log(R0))/(2*delta*PD)
zigma0=zigmaxmax/((order+1)*(PD^order)*2^(order+1))
for i=1:1:M-1
    for j=1:1:N-1
        for k=1:1:P-1

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if (1<=i && i<=M) && (1<=j && j<=N) && (1<=k && k<=P)

ep(i,j,k)=1;
end

if i<=PD+1
    mm=(PD-i)+1;
    if mm==0
        zigmax(i,j,k)=zigma0;
    end
    if mm~=0
        zigmax(i,j,k)=zigma0*(((2*mm+1)^(order+1)-(2*mm-1)^(order+1));
    mm=mm-0.5;
    zigmaxp(i,j,k)=zigma0*(((2*mm+1)^(order+1)-(2*mm-1)^(order+1));
    zigmaSx(i,j,k)=(mio0/ep0)*zigmaxp(i,j,k);
end

if i>=M-PD
    mm=(i-(M-PD));
    if mm==0
        zigmax(i,j,k)=zigma0;
    end
    if mm~=0
        zigmax(i,j,k)=zigma0*(((2*mm+1)^(order+1)-(2*mm-1)^(order+1));
    end
    if mm<PD
        mm=mm+0.5;
        zigmaxp(i,j,k)=zigma0*(((2*mm+1)^(order+1)-(2*mm-1)^(order+1));
        zigmaSx(i,j,k)=(mio0/ep0)*zigmaxp(i,j,k);
end

if j<=PD+1
    nn=(PD-j)+1;
    if nn==0
        zigmay(i,j,k)=zigma0;
    end
    if nn~=0
        zigmay(i,j,k)=zigma0*(((2*nn+1)^(order+1)-(2*nn-1)^(order+1));
    nn=nn-0.5;
    zigmayp(i,j,k)=zigma0*(((2*nn+1)^(order+1)-(2*nn-1)^(order+1));
    zigmaSy(i,j,k)=(mio0/ep0)*zigmayp(i,j,k);
end

if j>=N-PD
    nn=(j-(M-PD));
    if nn==0
        zigmay(i,j,k)=zigma0;
    end
    if nn~=0
        zigmay(i,j,k)=zigma0*(((2*nn+1)^(order+1)-(2*nn-1)^(order+1));
    end
    if nn<PD
nn=nn+0.5;
zigmayp(i,j,k)=zigma0*((2*nn+1)^((order+1)-1)1-(2*nn-1)^(order+1));
zigmaSy(i,j,k)=(mio0/ep0)*zigmayp(i,j,k);
end
if k<=PD+1
  kk=(PD-k)+1;
  if kk==0
    zigmaz(i,j,k)=zigma0;
  end
  if kk==0
    zigmaz(i,j,k)=zigma0*((2*kk+1)^((order+1)-1)1-(2*kk-1)^(order+1));
  kk=kk-0.5;
zigmazp(i,j,k)=zigma0*((2*kk+1)^((order+1)-1)1-(2*kk-1)^(order+1));
zigmaSz(i,j,k)=(mio0/ep0)*zigmazp(i,j,k);
end
if k>=P
  kk=(k-(P-PD));
  if kk==0
    zigmaz(i,j,k)=zigma0;
  end
  if kk==0
    zigmaz(i,j,k)=zigma0*((2*kk+1)^((order+1)-1)1-(2*kk-1)^(order+1));
  end
  if kk<PD
    kk=kk+0.5;
zigmazp(i,j,k)=zigma0*((2*kk+1)^((order+1)-1)1-(2*kk-1)^(order+1));
zigmaSz(i,j,k)=(mio0/ep0)*zigmazp(i,j,k);
end
% applying PML to the 6 sides
if i<=PD+1 && k>PD+1 && k<P-PD && j<N-PD && j>PD+1
  mm=(PD-i+1)+1/2;
zigmamax=(mm+1)/(150*pi*delta);
zigmaSx(i,j,k)=(mio0/ep0)*zigmamax(i,j,k);
zigmay(i,j,k)=0;
zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
zigmaz(i,j,k)=0;
zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end
if i>=M-PD && k>P-PD && k<P-PD && j<N-PD && j>PD+1
  mm=(i-(M-PD))1/2;
zigmamax=(mm+1)/(150*pi*delta);
zigmaSx(i,j,k)=(mio0/ep0)*zigmamax(i,j,k);
zigmay(i,j,k)=0;
zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
zigmaz(i,j,k)=0;
zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
if j<=PD+1 & k>PD+1 & k<P-PD & i=M-PD & i>PD+1
   nn=(PD-j+1)+1/2;
   zigmamax=(nn+1)/(150*pi*delta);
   zigmay(i,j,k)=(zigmamax*(nn/PD)^4);
   zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k); 
   zigmax(i,j,k)=0;
   zigmaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
   zigmaz(i,j,k)=0;
   zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if j>=N-PD & k>PD+1 & k<P-PD & i<M-PD & i>PD+1 
   nn=(j-(N-PD))+1/2;
   zigmamax=(nn+1)/(150*pi*delta);
   zigmay(i,j,k)=(zigmamax*(nn/PD)^4);
   zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
   zigmax(i,j,k)=0;
   zigmaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
   zigmaz(i,j,k)=0;
   zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if k<=PD+1 & j>PD+1 & j<N-PD & i<M-PD & i>PD+1 
   kk=(PD-k+1)+1/2;
   zigmamax=(kk+1)/(150*pi*delta);
   zigmaz(i,j,k)=(zigmamax*(kk/PD)^4);
   zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
   zigmax(i,j,k)=0;
   zigmaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
   zigmay(i,j,k)=0;
   zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
end

if k>=P-PD & j>PD+1 & j<N-PD & i<M-PD & i>PD+1
   kk=(k-(P-PD))+1/2;
   zigmamax=(kk+1)/(150*pi*delta);
   zigmaz(i,j,k)=(zigmamax*(kk/PD)^4);
   zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
   zigmax(i,j,k)=0;
   zigmaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
   zigmay(i,j,k)=0;
   zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
end

% applying PML to the 12 edges
% Bottom Edges
if k<=PD+1 & i<=PD+1 & j>PD+1 & j<N-PD
   mm=(PD-i+1)+1/2;
   zigmamax=(mm+1)/(150*pi*delta);
   zigmaz(i,j,k)=(zigmamax*(mm/PD)^4);
   zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
   kk=(PD-k+1)+1/2;
   zigmamax=(kk+1)/(150*pi*delta);
   zigmaz(i,j,k)=(zigmamax*(kk/PD)^4);
   zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if k<=PD+1 & j<=PD+1 & i>PD+1 & i<M-PD
   nn=(PD-j+1)+1/2;
% zigmamax = (nn+1)/(150*pi*delta);
%    zigmay(i,j,k) = zigmamax * (nn/PD)^4;
%    zigmaSy(i,j,k) = (mio0/ep0) * zigmay(i,j,k);
%    kk = (PD-k+1)+1/2;
% zigmamax = (kk+1)/(150*pi*delta);
%    zigmax(i,j,k) = zigmamax * (kk/PD)^4;
%    zigmaSz(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
%    zigmax(i,j,k) = 0;
%    zigmaSx(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
% end
%
% if k <= PD+1 && i >= M-PD && j > PD+1 && j < N-PD
%     mm = (i-(M-PD))+1/2;
%     zigmamax = (mm+1)/(150*pi*delta);
%     zigmax(i,j,k) = zigmamax * (mm/PD)^4;
%     zigmaSx(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
%     kk = (PD-k+1)+1/2;
% zigmamax = (kk+1)/(150*pi*delta);
%    zigmax(i,j,k) = zigmamax * (kk/PD)^4;
%    zigmaSz(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
%    zigmax(i,j,k) = 0;
%    zigmaSy(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
% end
%
% if k <= PD+1 && j >= N-PD && i > PD+1 && i < M-PD
%     nn = (j-(N-PD))+1/2;
%     zigmamax = (nn+1)/(150*pi*delta);
%     zigmay(i,j,k) = zigmamax * (nn/PD)^4;
%     zigmaSy(i,j,k) = (mio0/ep0) * zigmay(i,j,k);
%     kk = (PD-k+1)+1/2;
% zigmamax = (kk+1)/(150*pi*delta);
%    zigmax(i,j,k) = zigmamax * (kk/PD)^4;
%    zigmaSz(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
%    zigmay(i,j,k) = 0;
%    zigmaSx(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
% end
%
% if i <= PD+1 && j <= PD+1 && k > PD+1 && k < P-PD
%     mm = (PD-i+1)+1/2;
%     zigmamax = (mm+1)/(150*pi*delta);
%     zigmax(i,j,k) = zigmamax * (mm/PD)^4;
%     zigmaSx(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
%     nn = (PD-j+1)+1/2;
% zigmamax = (nn+1)/(150*pi*delta);
%    zigmay(i,j,k) = zigmamax * (nn/PD)^4;
%    zigmaSy(i,j,k) = (mio0/ep0) * zigmay(i,j,k);
%    zigmax(i,j,k) = 0;
%    zigmaSz(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
% end
%
% if i > = M-PD && j <= PD+1 && k > PD+1 && k < P-PD
%     mm = (i-(M-PD))+1/2;
%     zigmamax = (mm+1)/(150*pi*delta);
%     zigmax(i,j,k) = zigmamax * (mm/PD)^4;
%     zigmaSx(i,j,k) = (mio0/ep0) * zigmax(i,j,k);
%     nn = (PD-j+1)+1/2;
% zigmamax = (nn+1)/(150*pi*delta);
%    zigmay(i,j,k) = zigmamax * (nn/PD)^4;
%    zigmaSy(i,j,k) = (mio0/ep0) * zigmay(i,j,k);
%    zigmax(i,j,k) = 0;
% sigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if i>=M-PD && j>=N-PD && k>PD+1 && k<P-PD
  mm=(i-(M-PD))+1/2;
  zigmamax=(mm+1)/(150*pi*delta);
  zigmax(i,j,k)=zigmamax*(mm/PD)^4;
  zigaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
  nn=(j-(N-PD))+1/2;
  zigmamax=(nn+1)/(150*pi*delta);
  zigmay(i,j,k)=zigmamax*(nn/PD)^4;
  zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
  zigmaz(i,j,k)=0;
  zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if i<=PD+1 && j>=N-PD && k>PD+1 && k<P-PD
  mm=(PD-i+1)+1/2;
  zigmamax=(mm+1)/(150*pi*delta);
  zigmax(i,j,k)=zigmamax*(mm/PD)^4;
  zigaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
  nn=(j-(N-PD))+1/2;
  zigmamax=(nn+1)/(150*pi*delta);
  zigmay(i,j,k)=zigmamax*(nn/PD)^4;
  zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
  zigmaz(i,j,k)=0;
  zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

% Top edges
if k>=P-PD && i<=PD+1 && j>PD && j<N-PD
  mm=(PD-i+1)+1/2;
  zigmamax=(mm+1)/(150*pi*delta);
  zigmax(i,j,k)=zigmamax*(mm/PD)^4;
  zigaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
  kk=(k-(P-PD))+1/2;
  zigmamax=(kk+1)/(150*pi*delta);
  zigmay(i,j,k)=zigmamax*(kk/PD)^4;
  zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
  zigmaz(i,j,k)=0;
  zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if k>=P-PD && j<=PD+1 && i>PD && i<M-PD
  nn=(PD-j+1)+1/2;
  zigmamax=(nn+1)/(150*pi*delta);
  zigmay(i,j,k)=zigmamax*(nn/PD)^4;
  zigaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
  kk=(k-(P-PD))+1/2;
  zigmamax=(kk+1)/(150*pi*delta);
  zigmax(i,j,k)=zigmamax*(kk/PD)^4;
  zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
  zigmax(i,j,k)=0;
  zigaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
end

if k>=P-PD && i>=M-PD && j>PD+1 && j<N-PD
  mm=(i-(M-PD))+1/2;
  zigmamax=(mm+1)/(150*pi*delta);
  zigmax(i,j,k)=zigmamax*(mm/PD)^4;
  zigaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
  kk=(k-(P-PD))+1/2;
  zigmamax=(kk+1)/(150*pi*delta);
  zigmay(i,j,k)=zigmamax*(kk/PD)^4;
  zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
  zigmaz(i,j,k)=0;
  zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

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% zigmaz(i,j,k)=zigmamax*(kk/PD)^4;
% zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
% zigmay(i,j,k)=0;
% zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
end

if k>=P-PD && j>=N-PD && i>PD && i<M-PD
    nn=(j-(N-PD))+1/2;
    zigmamax=(nn+1)/(150*pi*delta);
    zigmay(i,j,k)=zigmamax*(nn/PD)^4;
    zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
    kk=(k-(P-PD))+1/2;
    zigmamax=(kk+1)/(150*pi*delta);
    zigmaz(i,j,k)=zigmamax*(kk/PD)^4;
    zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

% Applying corner PML Conditions to 8 corners

if i<=PD+1 && j<=PD+1 && k<=PD+1
    mm=(PD-i+1)+1/2;
    zigmamax=(mm+1)/(150*pi*delta);
    zigmax(i,j,k)=zigmamax*(mm/PD)^4;
    zigmaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
    nn=(PD-j+1)+1/2;
    zigmamax=(nn+1)/(150*pi*delta);
    zigmay(i,j,k)=zigmamax*(nn/PD)^4;
    zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
    kk=(PD-k+1)+1/2;
    zigmamax=(kk+1)/(150*pi*delta);
    zigmaz(i,j,k)=zigmamax*(kk/PD)^4;
    zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if i>=M-PD && j<=PD+1 && k<=PD+1
    mm=(i-(M-PD))+1/2;
    zigmamax=(mm+1)/(150*pi*delta);
    zigmax(i,j,k)=zigmamax*(mm/PD)^4;
    zigmaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
    nn=(PD-j+1)+1/2;
    zigmamax=(nn+1)/(150*pi*delta);
    zigmay(i,j,k)=zigmamax*(nn/PD)^4;
    zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
    kk=(PD-k+1)+1/2;
    zigmamax=(kk+1)/(150*pi*delta);
    zigmaz(i,j,k)=zigmamax*(kk/PD)^4;
    zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if i>=M-PD && j>=N-PD && k<=PD+1
    mm=(i-(M-PD)+1/2);
    zigmamax=(mm+1)/(150*pi*delta);
    zigmax(i,j,k)=zigmamax*(mm/PD)^4;
    zigmaSx(i,j,k)=(mio0/ep0)*zigmax(i,j,k);
    nn=(j-(N-PD))+1/2;
    zigmamax=(nn+1)/(150*pi*delta);
    zigmay(i,j,k)=zigmamax*(nn/PD)^4;
    zigmaSy(i,j,k)=(mio0/ep0)*zigmay(i,j,k);
    kk=(PD-k+1)+1/2;
    zigmamax=(kk+1)/(150*pi*delta);
    zigmaz(i,j,k)=zigmamax*(kk/PD)^4;
    zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end
\[
\text{zigmamax} = \frac{(kk+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmax}(i,j,k) = \text{zigmamax} \times (kk/PD)^4;
\]
\[
\text{zigmaSz}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{end}
\]

\[
\text{if } i \leq PD+1 \text{ \&\& } j = N-PD \text{ \&\& } k \leq PD+1
\]
\[
\text{mm} = (PD-i+1)+1/2;
\]
\[
\text{zigmamax} = \frac{(mm+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmax}(i,j,k) = \text{zigmamax} \times (mm/PD)^4;
\]
\[
\text{zigmaSx}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{nn} = (j-(N-PD))+1/2;
\]
\[
\text{zigmamax} = \frac{(nn+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmay}(i,j,k) = \text{zigmamax} \times (nn/PD)^4;
\]
\[
\text{zigmaSy}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmay}(i,j,k);
\]
\[
\text{kk} = (PD-k+1)+1/2;
\]
\[
\text{zigmamax} = \frac{(kk+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmaSz}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{endif}
\]

\[
\text{%% 4 top corners}
\]
\[
\text{if } i \leq PD+1 \text{ \&\& } j \leq PD+1 \text{ \&\& } k \geq PD+1
\]
\[
\text{mm} = (PD-i+1)+1/2;
\]
\[
\text{zigmamax} = \frac{(mm+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmax}(i,j,k) = \text{zigmamax} \times (mm/PD)^4;
\]
\[
\text{zigmaSx}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{nn} = (PD-j+1)+1/2;
\]
\[
\text{zigmamax} = \frac{(nn+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmay}(i,j,k) = \text{zigmamax} \times (nn/PD)^4;
\]
\[
\text{zigmaSy}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmay}(i,j,k);
\]
\[
\text{kk} = (k-(P-PD))+1/2;
\]
\[
\text{zigmamax} = \frac{(kk+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmaSz}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{endif}
\]

\[
\text{if } i \geq M-PD \text{ \&\& } j \leq PD+1 \text{ \&\& } k \geq PD+1
\]
\[
\text{mm} = (i-(M-PD))+1/2;
\]
\[
\text{zigmamax} = \frac{(mm+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmax}(i,j,k) = \text{zigmamax} \times (mm/PD)^4;
\]
\[
\text{zigmaSx}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{nn} = (PD-j+1)+1/2;
\]
\[
\text{zigmamax} = \frac{(nn+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmay}(i,j,k) = \text{zigmamax} \times (nn/PD)^4;
\]
\[
\text{zigmaSy}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmay}(i,j,k);
\]
\[
\text{kk} = (k-(P-PD))+1/2;
\]
\[
\text{zigmamax} = \frac{(kk+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmaSz}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{end}
\]

\[
\text{if } i \geq M-PD \text{ \&\& } j \geq N-PD \text{ \&\& } k \geq PD+1
\]
\[
\text{mm} = (i-(M-PD))+1/2;
\]
\[
\text{zigmamax} = \frac{(mm+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmax}(i,j,k) = \text{zigmamax} \times (mm/PD)^4;
\]
\[
\text{zigmaSx}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
\[
\text{nn} = (j-(N-PD))+1/2;
\]
\[
\text{zigmamax} = \frac{(nn+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmay}(i,j,k) = \text{zigmamax} \times (nn/PD)^4;
\]
\[
\text{zigmaSy}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmay}(i,j,k);
\]
\[
\text{kk} = (k-(P-PD))+1/2;
\]
\[
\text{zigmamax} = \frac{(kk+1)}{(150*\pi*\delta)};
\]
\[
\text{zigmaSz}(i,j,k) = \frac{(mio0/ep0)}{\epsilon_0} \times \text{zigmax}(i,j,k);
\]
% zigmamax=(kk+1)/(150*pi*delta);
% zigmaz(i,j,k)=zigmamax*(kk/PD)^4;
% zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

if i<=PD+1 && j>=N-PD && k>=P-PD
  mm=(PD-i+1)+1/2;
  zigmamax=(mm+1)/(150*pi*delta);
  zigmaSx(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
nn=(j-(N-PD))+1/2;
  zigmamax=(nn+1)/(150*pi*delta);
  zigmaSy(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
  kk=(k-(P-PD))+1/2;
  zigmamax=(kk+1)/(150*pi*delta);
  zigmaSz(i,j,k)=(mio0/ep0)*zigmaz(i,j,k);
end

Cae(i,j,k)=(1-Re*zigma(i,j,k)/ep(i,j,k))/(1+Re*zigma(i,j,k)/ep(i,j,k));
Cbe(i,j,k)=(Ra/ep(i,j,k))/(1+Re*zigma(i,j,k)/ep(i,j,k));
Cah(i,j,k)=(1-Rh*zigmam(i,j,k)/mio(i,j,k))/(1+Rh*zigmam(i,j,k)/mio(i,j,k));
Cbh(i,j,k)=(Rb/mio(i,j,k))/(1+Rh*zigmam(i,j,k)/mio(i,j,k));
CaeD(i,j,k)=(1-ReD*zigma(i,j,k)+ReD2)/(1+ReD*zigma(i,j,k)+ReD2);  %%%ReD=deltat/(2*epi*ep0);
CbeD(i,j,k)= RaD2/(1+ReD*zigma(i,j,k)+ReD2);
end

Caer=1;   Cber=(deltat/(ep0*delta));
Cahr=1;   Cbhr=deltat/(mio0*delta);

% fourier analysis of incident waveform
epsi=pi/2;
i0=PD+4;  j0=PD+4;  k0=PD+4;
i1=N-i0+1;  j1=N-j0+1;  k1=P-k0+1;
m0=20;
xm=2;  ym=37;  zm=0;
phi=atan(my/mx);  theta=pi/2;
kinc=[sin(theta)*cos(phi) sin(theta)*sin(phi) cos(theta)];
rcompmax=[i1-i0  j1-j0  k1-k0];
deltar=kinc(2)*delta/my;

ar0=[i0  j0  k0]*[mx  my  mz]';
amax=[i1  j1  k1]*[mx  my  mz]'*2;
Hx=zeros(arax,3);
Hy=zeros(army,3);
Hz=zeros(armz,3);
Ex=zeros(arax,3);
Ey=zeros(army,3);
Ez=zeros(armz,3);
mmax=max([abs(mx),abs(my),abs(mz)]);
imxmax=0;  imymax=0;  imzmax=0;
if mmax==mx;
imxmax=1;
end
if mmax==my;
imymax=1;
end
if mmax==mz;
imzmax=1;
end
summ=mx+my+mz;
summmaxH=2*(i1-1)*mx+2*(j1-1)*my+2*(k1-1)*mz;
nmax=3500

%%% starting time loop %%%

for n=1:1:nmax
    np=(n-1)*deltat
    pe=(10^(-5))*(-2*(np-3*T)/(T^2))*exp(-((np-3*T)/T)^2);

%%% End of connecting conditions %%%

arref=[i0-2 j0-2 k0-2];
sref=arref*[mx my mz]';
for irr=0:1:mmax-1
    npp=np-(irr)*deltar/c;
    Erx(sref+irr+my+mz,2)=0;
end
for irr=0:1:mmax-1
    npp=np-(irr)*deltar/c;
    Ery(sref+irr,2)=0;
end
for irr=0:1:mmax-1
    npp=np-(irr)*deltar/c;
    Erz(sref+irr,2)=(10^(-5))*(-2*(npp-3*T)/(T^2))*exp(-((npp-3*T)/T)^2)+Caer*Erz(sref+irr,2)+(Cber)*(Hry(sref+irr,2)-Hry(sref+irr-mx,2)-Hrx(sref+irr-my,2)-Hrx(sref+irr,2));
end

%%% updating E-fields on 1-D auxiliary grid %%%

for i=1:1:2*(i1-1)+1
    for j=1:1:2*(j1-1)+1
        for k=1:1:2*(k1-1)+1
            r=mx*i+my*j+mz*k;
            Erx(r,3)=Caer*Erx(r,2)+(Cber)*-(Hrz(r,2)-Hrz(r-my,2)+Hry(r-mz,2)-Hry(r,2));
            r=mx*i+my*j+mz*k;
            Ery(r,3)=Caer*Ery(r,2)+(Cber)*-(Hrx(r,2)-Hrx(r-mx,2)+Hrz(r-my,2)-Hrz(r,2));
            r=mx*i+my*j+mz*k;
            Erz(r,3)=Caer*Erz(r,2)+(Cber)*-(Hry(r,2)-Hry(r-mx,2)+Hrx(r-my,2)-Hrx(r,2));
        end
    end
end

%%% updating H-fields on 1-D auxiliary grid %%%

for ih=0:1:1
for jh=0:1:1
    for kh=0:1:1
        Ar1=ih*2^2+jh*2+kh+1;
        Ar2=mx*ih+my*jh+mz*kh;
        rcxl(Ar1)=Hrx(summ+Ar2,3);
        rcy(Ar1)=Hry(summ+Ar2,3);
        rcz(Ar1)=Hrz(summ+Ar2,3);
    end
end
end

for ih=0:1:1
    for jh=0:1:1
        for kh=0:1:1
            Ar1=ih*2^2+jh*2+kh+1;
            Ar2=mx*ih+my*jh+mz*kh;
            rccx(Ar1)=Hrx(summaxH-Ar2,3);
            rccy(Ar1)=Hry(summaxH-Ar2,3);
            rccz(Ar1)=Hrz(summaxH-Ar2,3);
        end
    end
end

for i=1:1:2*(i1-1)                    % from 1-imxmax
    for j=1:1:2*(j1-1)                % 1-imymax
        for k=1:1:2*(k1-1)            % k=1-imzmax
            r=mx*i+my*j+mz*k; rl=r;
            Hrx(r,3)=Cahr*Hrx(r,2)+Cbhr*(Ery(r+mz,3)-Ery(r,3)+Erz(r,3)-Erz(r+mz,3));
            r=mx*i+my*j+mz*k; rl=r;
            Hry(r,3)=Cahr*Hry(r,2)+Cbhr*(Erz(r+mx,3)-Erz(r,3)+Erx(r,3)-Erx(r+mx,3));
            r=mx*i+my*j+mz*k; rl=r;
            Hrz(r,3)=Cahr*Hrz(r,2)+Cbhr*(Erx(r+my,3)-Erx(r,3)+Ery(r,3)-Ery(r+my,3));
        end
    end
end
end

\%
\%
% Run this section if using interpolation method
\%
% Hrx(r,3)=Cah(i,j,k)*Hrx(r,2)+Cbh(i,j,k)*(Ery(r+1,2)-Ery(r,2)+Erz(r,2)-Erz(r+1,2));
% \% Hry(r,3)=Cah(i,j,k)*Hry(r,2)+Cbh(i,j,k)*(Erz(r+1,2)-Erz(r,2)+Erx(r,2)-Erx(r+1,2));
% % Hrz(r,3)=Cah(i,j,k)*Hrz(r,2)+Cbh(i,j,k)*(Erx(r+1,2)-Erx(r,2)+Ery(r,2)-Ery(r+1,2));
% % Hrx(r,3)=Hrx(r,3)*(sin(epsilon)*sin(phi)+cos(epsilon)*cos(theta)*cos(phi));
% % Hry(r,3)=Hry(r,3)*(-sin(epsilon)*cos(phi)+cos(epsilon)*cos(theta)*sin(phi));
% % Hrz(r,3)=Hrz(r,3)*(sin(epsilon)*sin(theta));
% end
% \%
% rcomp=[i-i0 j-j0 k-k0];

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% d=kinc*rcomp';
% dp=d-fix(d);
%
% if i==i0 || j==j0 || k==k0
% Einc=(1-dp)*Einc1(m0+fix(d),3)+dp*Einc1(m0+fix(d)+1,3);
% % Hinc=(1-dp)*Hinc1(m0+fix(d),3);
% end
%
% if i==i1 || j==j1 || k==k1
% Einc=(1-dp)*Einc1(m0+fix(d),3)+dp*Einc1(m0+fix(d)+1,3);
% % Hinc=(1-dp)*Hinc1(m0+fix(d),3)+dp*Hinc1(m0+fix(d),3);
% end
%
% dz=d+1;
% dzp=dz-fix(d);
% if i==i0-1
% rcomp=[i-1 i-1 j-1 k-1];
% d=kinc*rcomp';
% dp=d-fix(d);
% Einc=(1-dp)*Einc1(m0+fix(d),3)+dp*Einc1(m0+fix(d)+1,3);
end
%
% if j==j0-1
% rcomp=[i-1 i-1 j-1 k-1];
% d=kinc*rcomp';
% dp=d-fix(d);
% Einc=(1-dp)*Einc1(m0+fix(d),3)+dp*Einc1(m0+fix(d)+1,3);
end
%
% if k==k0-1
% rcomp=[i-1 i-1 j-1 k-1];
% d=kinc*rcomp';
% dp=d-fix(d);
% Einc=(1-dp)*Einc1(m0+fix(d),3)+dp*Einc1(m0+fix(d)+1,3);
end
%
% if i==i1 || j==j1 || k==k1
% Einc=(1-dp)*Hinc1(m0-1+fix(d),3)+dp*Hinc1(m0+fix(d)+1,3);
% end
%
% if i==i0-1
% Hyinc(i-1,j,k)=Hinc*(-sin(epsilon)*cos(phi)+cos(epsilon)*cos(theta)*sin(phi));
% Hzinc(i-1,j,k)=Hinc*(-cos(epsilon)*sin(theta));
% end
%
% if i==i0
%  Eyinc(i0,j,k)=Einc*(-cos(eps)*cos(phi)-
%                   sin(eps)*cos(theta)*sin(phi));
%  Ezinc(i0,j,k)=Einc*(sin(eps)*sin(theta));
%  end
%  if j==j0-1
%     Hxinc(i,j0-1,k)=Hinc*(sin(eps)*sin(phi)+cos(eps)*cos(theta)*cos(phi));
%     Hzinc(i,j0-1,k)=Hinc*(-cos(eps)*sin(theta));
%  end
%  if j==j0
%     Exinc(i,j0,k)=Einc*(cos(eps)*sin(phi)-
%                      sin(eps)*cos(theta)*cos(phi));
%     Ezinc(i,j0,k)=Einc*(sin(eps)*sin(theta));
%  end
%  if k==k0-1
%     Hxinc(i,j,k0-1)=Hinc*(sin(eps)*sin(phi)+cos(eps)*cos(theta)*cos(phi));
%     Hyinc(i,j,k0-1)=Hinc*(-sin(eps)*cos(phi)+cos(eps)*cos(theta)*sin(phi));
%     Eyinc(i,j,k0)=Einc*(-cos(eps)*cos(phi)-
%                        sin(eps)*cos(theta)*sin(phi));
%     Ezinc(i,j,k0)=Einc*(sin(eps)*sin(theta));
%  end
%  if i==i1
%      Hyinc(i1,j,k)=Hinc*(-sin(eps)*cos(phi)+cos(eps)*cos(theta)*sin(phi));
%      Hzinc(i1,j,k)=Hinc*(-cos(eps)*sin(theta));
%      Eyinc(i1,j,k)=Einc*(cos(eps)*sin(phi)-
%                        sin(eps)*cos(theta)*cos(phi));
%      Ezinc(i1,j,k)=Einc*(sin(eps)*sin(theta));
%  end
%  if j==j1
%      Hxinc(i,j1,k)=Hinc*(sin(eps)*sin(phi)+cos(eps)*cos(theta)*cos(phi));
%      Hzinc(i,j1,k)=Hinc*(-cos(eps)*sin(theta));
%      Exinc(i,j1,k)=Einc*(cos(eps)*sin(phi)-
%                        sin(eps)*cos(theta)*cos(phi));
%      Ezinc(i,j1,k)=Einc*(sin(eps)*sin(theta));
%  end
%  if k==k1
%      Hxinc(i,j,k1)=Hinc*(sin(eps)*sin(phi)+cos(eps)*cos(theta)*cos(phi));
%      Hyinc(i,j,k1)=Hinc*(-cos(eps)*sin(theta));
%      Exinc(i,j,k1)=Einc*(cos(eps)*sin(phi)-
%                        sin(eps)*cos(theta)*cos(phi));
%      Eyinc(i,j,k1)=Einc*(-cos(eps)*cos(phi)-
%                        sin(eps)*cos(theta)*sin(phi));
%  end
%  end
% end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
for ih=0:1:1
for jh=0:1:1
    for kh=0:1:1
        Ar1=ih*2^2+jh*2+kh+1;
        Ar2=mx*ih+my*jh+mz*kh;
        if summ+Ar2-my<summ
            Hrx(summ+Ar2-my,3)=rbclx(Ar1);
        end
        if summ+Ar2-mz<summ
            Hrx(summ+Ar2-mz,3)=rbclx(Ar1);
        end
        if summ+Ar2-mx<summ
            Hry(summ+Ar2-mx,3)=rbcly(Ar1);
        end
        if summ+Ar2-mz<summ
            Hry(summ+Ar2-mz,3)=rbcly(Ar1);
        end
        if summ+Ar2-mx<summ
            Hrz(summ+Ar2-mx,3)=rbclz(Ar1);
        end
        if summ+Ar2-my<summ
            Hrz(summ+Ar2-my,3)=rbclz(Ar1);
        end
    end
end
for ih=0:1:1
    for jh=0:1:1
        for kh=0:1:1
            Ar1=ih*2^2+jh*2+kh+1;
            Ar2=mx*ih+my*jh+mz*kh;
            if summmaxH-Ar2+my>summmaxH
                Hrx(summmaxH-Ar2+my,3)=rbcx(Ar1);
            end
            if summmaxH-Ar2+mz>summmaxH
                Hrx(summmaxH-Ar2+mz,3)=rbcx(Ar1);
            end
            if summmaxH-Ar2+mx>summmaxH
                Hry(summmaxH-Ar2+mx,3)=rbcy(Ar1);
            end
            if summmaxH-Ar2+mz>summmaxH
                Hry(summmaxH-Ar2+mz,3)=rbcy(Ar1);
            end
            if summmaxH-Ar2+mx>summmaxH
                Hrz(summmaxH-Ar2+mx,3)=rbcz(Ar1);
            end
            if summmaxH-Ar2+my>summmaxH
                Hrz(summmaxH-Ar2+my,3)=rbcz(Ar1);
            end
        end
    end
end
%Applying outer layer PEC boundary conditions
for i=1:1:M
    for j=1:1:N
        for k=1:1:P
            if i==1 || i==M
                % Code for PEC boundary conditions
            end
        end
    end
end
Ey(i,j,k,2)=0;
Eyz(i,j,k,2)=0;
Ezx(i,j,k,2)=0;
Ezy(i,j,k,2)=0;
end
if j==1 || j==N
Exy(i,j,k,2)=0;
Exz(i,j,k,2)=0;
Eyx(i,j,k,2)=0;
Eyz(i,j,k,2)=0;
end
if k==1 || k==P
Exy(i,j,k,2)=0;
Exz(i,j,k,2)=0;
Eyx(i,j,k,2)=0;
Eyz(i,j,k,2)=0;
end
end
end
end

%update magnetic fields
for i=1:1:M-1
  for j=1:1:N-1
    for k=1:1:P-1
      if (i>=PD+1 && i<=M-PD-1) && (j>=PD+1 && j<=N-PD-1) && (k>=PD+1 && k<=P-PD-1)
        Hx(i,j,k,3)=Cah(i,j,k)*Hx(i,j,k,2)+Cbh(i,j,k)*(Ey(i,j,k+1,2)-Ey(i,j,k,2)+Ez(i,j,k,2)-Ez(i,j+1,k,2));
        Hy(i,j,k,3)=Cah(i,j,k)*Hy(i,j,k,2)+Cbh(i,j,k)*(Ez(i+1,j,k,2)-Ez(i,j,k,2)+Ex(i,j,k,2)-Ex(i,j,k+1,2));
        Hz(i,j,k,3)=Cah(i,j,k)*Hz(i,j,k,2)+Cbh(i,j,k)*(Ex(i,j+1,k,2)-Ex(i,j,k,2)+Ey(i,j,k,2)-Ey(i+1,j,k,2));
      end
      rtest=([i j k])*[mx my mz]';
      if j==j0-1 && (i>=i0 && i<=i1) && (k>=k0 && k<=k1)
        if i==i1
          Hz(i,j0-1,k,3)=Hz(i,j0-1,k,3)-Cbh(i,j0-1,k)*Erz(rtest+my,3);
        end
        if k==k1
          Hx(i,j0-1,k,3)=Hx(i,j0-1,k,3)+Cbh(i,j0-1,k)*Erz(rtest+my,3);
        end
      end
      if j==j1 && (i>=i0 && i<=i1) && (k>=k0 && k<=k1)
        if i==i1
          Hz(i,j1,k,3)=Hz(i,j1,k,3)+Cbh(i,j1,k)*Erz(rtest,3);
        end
      end
      end
    end
  end
end
end
if k~=k1
  Hx(i,j1,k,3)=Hx(i,j1,k,3)-Cbh(i,j1,k)*Erz(rtest,3);
end
endif

if k==k0-1 && (i>=i0 && i<=i1) && (j>=j0 && j<=j1)
  if i~i1
    Hy(i,j,k0-1,3) = Hy(i,j,k0-1,3)+Cbh(i,j,k0-1)*Erx(rtest+mz,3);
  endif
  if j~j1
    Hx(i,j,k0-1,3) = Hx(i,j,k0-1,3)-Cbh(i,j,k0-1)*Ery(rtest+mz,3);
  endif
end
endif

if k==k1 && (i>=i0 && i<=i1) && (j>=j0 && j<=j1)
  if i~i1
    Hy(i,j,k1,3)= Hy(i,j,k1,3)+Cbh(i,j,k1)*Erx(rtest,3);
  endif
  if j~j1
    Hx(i,j,k1,3) = Hx(i,j,k1,3)-Cbh(i,j,k1)*Ery(rtest,3);
  endif
end
endif

%%% End of connecting conditions %%%

%Updating magnetic fields on the vacuum-PML interface
%starting from 2 since the outer most layer is conductive

if i<PD+1 && k>=PD+1 && k<P-PD && j<N-PD && j>=PD+1
  Hxy(i,j,k,3)=Hxy(i,j,k,2)-
    ((deltat/(mio0*mio(i,j,k)))/delta)*(Ezx(i,j+1,k,2)+Ezy(i,j+1,k,2)-
    Ezx(i,j,k,2)-Ezy(i,j,k,2));
endif

Hxz(i,j,k,3)=Hxz(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/delta)*(Eyz(i,j,k+1,2)+Eyx(i,j,k+1,2)-
    Eyz(i,j,k,2)-Eyx(i,j,k,2));

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Hyz(i,j,k,3)=Hyz(i,j,k,2)-
((deltat/(mio0*mio(i,j,k)))/delta)*(Exy(i,j,k+1,2)+Exz(i,j,k+1,2)-
Exy(i,j,k,2)-Exz(i,j,k,2));

Hxy(i,j,k,3)=exp(-
 sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2)+((1-exp(-
 sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSx(i,j,k)*delta)))*(Exz( i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

Hzx(i,j,k,3)=exp(-
 sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzx(i,j,k,2)-((1-exp(-
 sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSx(i,j,k)*delta)))*(Eyz( i+1,j,k,2)+Eyx(i+1,j,k,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));

Hzy(i,j,k,3)=Hzy(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/delta)*(Exy(i,j+1,
k,2)+Exz(i,j+1,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

end

if i>=M-PD && k>=PD && j<N-PD && j>=PD+1

Hxy(i,j,k,3)=Hxy(i,j,k,2)-
((deltat/(mio0*mio(i,j,k)))/delta)*(Exz(i,j,k+1,2)+Ezy(i,j,k+1,2)-
Ezx(i,j,k,2)-Ezy(i,j,k,2));

Hzx(i,j,k,3)=Hxz(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/delta)*(Eyz(i,j+1,
k,2)+Eyx(i,j,k+1,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));

Hzy(i,j,k,3)=Hzy(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/delta)*(Exy(i,j+1,
k,2)+Exz(i,j+1,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

end

if j<PD+1 && k>=PD+1 && i<M-PD && i>=PD+1

Hyz(i,j,k,3)=Hyz(i,j,k,2)-
((deltat/(mio0*mio(i,j,k)))/delta)*(Exy(i,j,k+1,2)+Exz(i,j,k+1,2)-
Exy(i,j,k,2)-Exz(i,j,k,2));

Hxy(i,j,k,3)=exp(-
 sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2)+((1-exp(-
 sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSy(i,j,k)*delta)))*(Ezx( i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

Hxz(i,j,k,3)=Hxz(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/delta)*(Eyz(i,j,k+1,2)+Eyx(i,j,k+1,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));

end
Hzx(i,j,k,3) = Hzx(i,j,k,2) - ((deltat/(mio0*mio(i,j,k)))/delta) * (Eyz(i+1,j,k,2) + Eyx(i+1,j,k,2) - Eyz(i,j,k,2) - Eyx(i,j,k,2));

Hyz(i,j,k,3) = exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))) * Hyz(i,j,k,2) + ((1-exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSy(i,j,k)*delta)) * (Exy(i,j+1,k,2) + Exz(i,j+1,k,2) - Exy(i,j,k,2) - Exz(i,j,k,2));

end if

if j >= N-PD && k >= PD+1 && i < M-PD && i >= PD+1

Hyz(i,j,k,3) = Hzx(i,j,k,2) - ((deltat/(mio0*mio(i,j,k)))/delta) * (Eyz(i,j,k+1,2) + Eyx(i,j,k+1,2) - Eyz(i,j,k,2) - Eyx(i,j,k,2));

Hyz(i,j,k,3) = Hzx(i,j,k,3) - ((deltat/(mio0*mio(i,j,k)))/delta) * (Exy(i,j,k)) - Exz(i,j,k,2) - Ezx(i,j,k,2) - Hxy(i,j,k,2);
Hxz(i,j,k,3)=Hzx(i,j,k,2)-((deltat/(mio0*mio(i,j,k)))/delta)*(Eyz(i+1,j,k,2)+Eyx(i+1,j,k,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));

Hzy(i,j,k,3)=Hzy(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/delta)*(Exy(i+1,j,k,2)+Exz(i+1,j,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hxy(i,j,k,3)=Hxy(i,j,k,2)-((deltat/(mio0*mio(i,j,k)))/delta)*(Ezx(i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

Hxz(i,j,k,3)=exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzx(i,j,k,2)+((1-exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSz(i,j,k)*delta))*(Eyz(i,j+1,k,2)+Eyx(i,j+1,k,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));

Hzy(i,j,k,3)=exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzy(i,j,k,2)+((1-exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSz(i,j,k)*delta))*(Exy(i,j,k+1,2)+Exz(i,j,k+1,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hxy(i,j,k,3)=exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2)+((1-exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSz(i,j,k)*delta))*(Ezx(i,j+1,k,2)+Ezy(i,j+1,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

Hyz(i,j,k,3)=exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2)+((1-exp(-zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSz(i,j,k)*delta))*(Exy(i+1,j,k,2)+Exz(i+1,j,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hyx(i,j,k,3)=exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2)+((1-exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta))*(Ezx(i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

end

% Applying PML Conditions to 12 edges

if k<PD+1 & & i<PD+1 & & j>=PD+1 & & j<N

Hxy(i,j,k,3)=exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2)-((deltat/(mio0*mio(i,j,k)))/delta)*(Ezx(i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

Hxz(i,j,k,3)=exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzx(i,j,k,2)+((1-exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta))*(Eyz(i+1,j,k,2)+Eyx(i+1,j,k,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));

Hyz(i,j,k,3)=exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2)-((1-exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta))*(Exy(i+1,j,k,2)+Exz(i+1,j,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hyx(i,j,k,3)=exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2)+((1-exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSy(i,j,k)*delta))*(Ezx(i,j+1,k,2)+Ezy(i,j+1,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

end

if k<PD+1 & & j<PD+1 & & i>=PD+1 & & i<M

Hyz(i,j,k,3)=Hyz(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/delta)*(Exy(i,j+1,k,2)+Exz(i,j+1,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hyx(i,j,k,3)=exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2)-((deltat/(mio0*mio(i,j,k)))/delta)*(Ezx(i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

end
\[ \text{Hxz}(i,j,k,3) = \exp(- \Sigma(i,j,k) \cdot \delta) \cdot \text{Hxz}(i,j,k,2) + (1-\exp(- \Sigma(i,j,k) \cdot \delta)) \cdot (\text{Eyz}(i,j,k+1,2) + \text{Eyx}(i,j,k+1,2) - \text{Eyz}(i,j,k,2) - \text{Eyx}(i,j,k,2)) \]

\[ \text{Hyz}(i,j,k,3) = \exp(- \Sigma(i,j,k) \cdot \delta) \cdot \text{Hyz}(i,j,k,2) - (1-\exp(- \Sigma(i,j,k) \cdot \delta)) \cdot (\text{Exy}(i,j,k+1,2) + \text{Exz}(i,j,k+1,2) - \text{Exy}(i,j,k,2) - \text{Exz}(i,j,k,2)) \]

\[ \text{Hyx}(i,j,k,3) = \exp(- \Sigma(i,j,k) \cdot \delta) \cdot \text{Hyx}(i,j,k,2) + (1-\exp(- \Sigma(i,j,k) \cdot \delta)) \cdot (\text{Ezx}(i,j,k+1,2) + \text{Ezy}(i,j,k+1,2) - \text{Ezx}(i,j,k,2) - \text{Ezy}(i,j,k,2)) \]

\[ \text{Hzx}(i,j,k,3) = \exp(- \Sigma(i,j,k) \cdot \delta) \cdot \text{Hzx}(i,j,k,2) - (1-\exp(- \Sigma(i,j,k) \cdot \delta)) \cdot (\text{Eyz}(i,j,k+1,2) + \text{Eyx}(i,j,k+1,2) - \text{Eyz}(i,j,k,2) - \text{Eyx}(i,j,k,2)) \]

\[ \text{Hxy}(i,j,k,3) = \exp(- \Sigma(i,j,k) \cdot \delta) \cdot \text{Hxy}(i,j,k,2) - (1-\exp(- \Sigma(i,j,k) \cdot \delta)) \cdot (\text{Exy}(i,j,k+1,2) + \text{Exz}(i,j,k+1,2) - \text{Exy}(i,j,k,2) - \text{Exz}(i,j,k,2)) \]
\begin{verbatim}
end

Hyz(i,j,k,3)=exp(-
  sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2)-((1-exp(-
  sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))/(sigmaSz(i,j,k)*deltat))\times(E
  xy(i,j,k+1,2)+Exz(i,j,k+1,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hyx(i,j,k,3)=Hyx(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/deltat)\times(Exy(i+1, j
  ,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

if  j~N

  Hzx(i,j,k,3)=Hzx(i,j,k,2)-
    ((deltat/(mio0*mio(i,j,k)))/deltat)\times(Ezy(i+1,j,k,2)+Eyx(i+1,j,k,2
  )-Ezx(i,j,k,2)-Ezy(i,j,k,2));
  Hzx(i,j,k,3)=exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzx(i,j,k,2)-((1-exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))/(sigmaSy(i,j,k)*deltat))\times(E
    xy(i+1,j+1,k,2)+Exz(i+1,j+1,k,2)-Ezx(i,j+1,k,2)-Ezy(i,j,k,2));

Hyz(i,j,k,3)=Hyz(i,j,k,2)-
  ((deltat/(mio0*mio(i,j,k)))/deltat)\times(Exy(i,j,k+1,2)+Exz(i,j,k+1,2
  )-Exy(i,j,k,2)-Exz(i,j,k,2));

Hyx(i,j,k,3)=exp(-
  sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2)-((1-exp(-
  sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))/(sigmaSx(i,j,k)*deltat))\times(E
  xy(i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

end

end
% side edges

if  i<PD+1 && j<PD+1 && k>=PD+1 && k<P

  Hzx(i,j,k,3)=Hzx(i,j,k,2)-
    ((deltat/(mio0*mio(i,j,k)))/deltat)\times(Ezy(i+1,j,k,2)+Eyx(i+1,j,k,2
  )-Ezx(i,j,k,2)-Ezy(i,j,k,2));
  Hzy(i,j,k,3)=exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzy(i,j,k,2)-(
    1-exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))/(sigmaSy(i,j,k)*deltat))\times(E
    xy(i,j+1,k,2)+Exz(i,j,k+1,2)-Ezy(i,j,k,2)-Eyx(i,j,k,2));

Hyz(i,j,k,3)=Hyz(i,j,k,2)-
  ((deltat/(mio0*mio(i,j,k)))/deltat)\times(Exy(i,j,k+1,2)+Exz(i,j,k+1,2
  )-Exy(i,j,k,2)-Exz(i,j,k,2));

end

if  i>=M-PD && j<PD+1 && k>=PD+1 && k<P

  Hxy(i,j,k,3)=exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2)-((1-exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))/(sigmaSy(i,j,k)*deltat))\times(E
    xy(i,j+1,k,2)+Exz(i,j,k+1,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

  Hxz(i,j,k,3)=Hxz(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/deltat)\times(E
    xy(i,j,k+1,2)+Ezy(i,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

Hyx(i,j,k,3)=exp(-
  sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2)-((1-exp(-
  sigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))/(sigmaSx(i,j,k)*deltat))\times(E
  xy(i+1,j,k,2)+Ezy(i+1,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

end

if  i>=M-PD && j<PD+1 && k>P-PD

  Hxy(i,j,k,3)=exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2)-((1-exp(-
    sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))/(sigmaSy(i,j,k)*deltat))\times(E
    xy(i,j+1,k,2)+Exz(i,j,k+1,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

  Hxz(i,j,k,3)=Hxz(i,j,k,2)+((deltat/(mio0*mio(i,j,k)))/deltat)\times(E
    xy(i,j,k+1,2)+Ezy(i,j,k,2)-Ezx(i,j,k,2)-Ezy(i,j,k,2));

end

end

end
\end{verbatim}
Hyx(i,j,k,3) = \exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2) + ((1-\exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta)) * (Ezx(i+1,j,k,2) + Ezy(i+1,j,k,2) - Ezx(i,j,k,2) - Ezy(i,j,k,2));

Hxz(i,j,k,3) = \exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxz(i,j,k,2) - ((1-\exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta)) * (Ezx(i+1,j,k,2) + Ezy(i+1,j,k,2) - Ezx(i,j,k,2) - Ezy(i,j,k,2));

Hzy(i,j,k,3) = \exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzy(i,j,k,2) + ((1-\exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSy(i,j,k)*delta)) * (Exy(i,j+1,k,2) + Exz(i,j+1,k,2) - Exy(i,j,k,2) - Exz(i,j,k,2));

end

if i>=M-PD && j>=N-PD && k>=PD+1 && k<P-PD

Hxy(i,j,k,3) = \exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2) - ((1-\exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSy(i,j,k)*delta)) * (Ezx(i+1,j,k,2) + Ezy(i+1,j,k,2) - Ezx(i,j,k,2) - Ezy(i,j,k,2));

Hxz(i,j,k,3) = Hxz(i,j,k,2) + ((deltat/(mio0*mio(i,j,k)))/delta) * (Exy(i,j,k+1,2) + Exz(i,j,k+1,2) - Exy(i,j,k,2) - Exz(i,j,k,2));

Hyz(i,j,k,3) = Hyz(i,j,k,2) - ((deltat/(mio0*mio(i,j,k)))/delta) * (Exy(i,j,k+1,2) + Exz(i,j,k+1,2) - Exy(i,j,k,2) - Exz(i,j,k,2));

Hyx(i,j,k,3) = \exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2) + ((1-\exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta)) * (Ezx(i+1,j,k,2) + Ezy(i+1,j,k,2) - Ezx(i,j,k,2) - Ezy(i,j,k,2));

Hzx(i,j,k,3) = \exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzx(i,j,k,2) - ((1-\exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta)) * (Ezy(i+1,j,k,2) + Eyx(i+1,j,k,2) - Ezy(i,j,k,2) - Eyx(i,j,k,2));

end

if i<P+1 && j>=N-PD && k>=PD+1 && k<P-PD

Hxy(i,j,k,3) = \exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2) - ((1-\exp(-zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSy(i,j,k)*delta)) * (Ezx(i+1,j,k,2) + Ezy(i+1,j,k,2) - Ezx(i,j,k,2) - Ezy(i,j,k,2));

Hxz(i,j,k,3) = Hxz(i,j,k,2) + ((deltat/(mio0*mio(i,j,k)))/delta) * (Exy(i,j,k+1,2) + Exz(i,j,k+1,2) - Exy(i,j,k,2) - Exz(i,j,k,2));

Hyz(i,j,k,3) = Hyz(i,j,k,2) - ((deltat/(mio0*mio(i,j,k)))/delta) * (Exy(i,j,k+1,2) + Exz(i,j,k+1,2) - Exy(i,j,k,2) - Exz(i,j,k,2));

Hyx(i,j,k,3) = \exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2) + ((1-\exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta)) * (Ezx(i+1,j,k,2) + Ezy(i+1,j,k,2) - Ezx(i,j,k,2) - Ezy(i,j,k,2));

Hzx(i,j,k,3) = \exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzx(i,j,k,2) - ((1-\exp(-zigmaSx(i,j,k)*deltat/(mio0*mio(i,j,k))))/(zigmaSx(i,j,k)*delta)) * (Ezy(i+1,j,k,2) + Eyx(i+1,j,k,2) - Ezy(i,j,k,2) - Eyx(i,j,k,2));
Hzy(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hzy(i,j,k,2) + ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Exy(i,j+1,k,2)} - \text{Eyz(i,j,k+1,2)}) + (\text{Exy(i,j+1,k,2)} - \text{Eyz(i,j,k,2)} - \text{Exz(i,j,k+1,2)} + \text{Eyz(i,j,k,2)}); 
end

if k>PD-1 && j<PD+1 && i<PD+1 

Hxy(i,j,k,3)=Hxy(i,j,k,2) - ((\text{deltat} / (\mu_0 \cdot \mu(i,j,k))) / (\text{delta})) \cdot (\text{Ezx(i,j+1,k,2)} + \text{Ezy(i,j,k+1,2)} - \text{Ezx(i,j,k,2)} - \text{Ezy(i,j,k,2)}); 
Hxz(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hxz(i,j,k,2) + ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Eyz(i,j+1,k,2)} - \text{Eyx(i,j,k+1,2)} - \text{Eyz(i,j,k,2)} - \text{Eyx(i,j,k,2)}); 
Hyz(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hyz(i,j,k,2) - ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Exy(i,j+1,k,2)} - \text{Eyz(i,j,k+1,2)} - \text{Exz(i,j,k+1,2)} + \text{Eyz(i,j,k,2)} - \text{Eyx(i,j,k,2)}); 

Hyz(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hyz(i,j,k,2) - ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Exy(i,j+1,k,2)} - \text{Eyz(i,j,k+1,2)} - \text{Exz(i,j,k+1,2)} + \text{Eyz(i,j,k,2)} - \text{Eyx(i,j,k,2)}); 

if k>PD-1 && j<PD+1 && i=M-PD 

Hxy(i,j,k,3)=Hxy(i,j,k,2) - ((\text{deltat} / (\mu_0 \cdot \mu(i,j,k))) / (\text{delta})) \cdot (\text{Exz(i,j+1,k,2)} - \text{Eyz(i,j,k+1,2)} - \text{Exz(i,j,k,2)} - \text{Eyz(i,j,k,2)}); 
Hxz(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hxz(i,j,k,2) + ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Eyz(i,j+1,k,2)} - \text{Eyx(i,j,k+1,2)} - \text{Eyz(i,j,k,2)} - \text{Eyx(i,j,k,2)}); 
Hyz(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hyz(i,j,k,2) - ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Exy(i,j+1,k,2)} - \text{Eyz(i,j,k+1,2)} - \text{Exz(i,j,k+1,2)} + \text{Eyz(i,j,k,2)} - \text{Eyx(i,j,k,2)}); 

if k>PD-1 && j=PD+1 && i=M-PD 

Hxy(i,j,k,3)=Hxy(i,j,k,2) - ((\text{deltat} / (\mu_0 \cdot \mu(i,j,k))) / (\text{delta})) \cdot (\text{Ezx(i,j+1,k,2)} + \text{Ezy(i,j,k+1,2)} - \text{Ezx(i,j,k,2)} - \text{Ezy(i,j,k,2)}); 
Hxz(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hxz(i,j,k,2) + ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Eyz(i,j+1,k,2)} - \text{Eyx(i,j,k+1,2)} - \text{Eyz(i,j,k,2)} - \text{Eyx(i,j,k,2)}); 
Hyz(i,j,k,3)=\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k))) \cdot Hyz(i,j,k,2) - ((1-\exp(-
\sigma_S(i,j,k) \cdot \text{deltat} / (\mu_0 \cdot \mu(i,j,k)))) / (\sigma_S(i,j,k) \cdot \text{delta})) \cdot (\text{Exy(i,j+1,k,2)} - \text{Eyz(i,j,k+1,2)} - \text{Exz(i,j,k+1,2)} + \text{Eyz(i,j,k,2)} - \text{Eyx(i,j,k,2)}); 

end
if k>=P-PD && i>=M-PD && j>=PD+1 && j<N-PD

Hxy(i,j,k,3)=Hxy(i,j,k,2) -
((deltat/(mio0*mio(i,j,k)))/delta)*(Ezx(i,j+1,k,2)+Ezy(i,j+1,k,2) -
Ezx(i,j,k,2)-Ezy(i,j,k,2));
Hxz(i,j,k,3)=exp(-
(zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxz(i,j,k,2)+((1-exp(-
zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSz(i,j,k)*delta)) * (Eyz(i,j+1,k,2)+Eyx(i,j+1,k,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));
Hyz(i,j,k,3)=exp(-
(zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2) -
((1-exp(-
zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSz(i,j,k)*delta)) * (Exy(i,j+1,k,2)+Exz(i,j+1,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hxy(i,j,k,3)=exp(-
(zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2) -
((1-exp(-
zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSy(i,j,k)*delta)) * (Ezx(i,j+1,k,2)+Ezy(i,j+1,k,2) -
Ezx(i,j,k,2)-Ezy(i,j,k,2));
Hxz(i,j,k,3)=exp(-
(zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxz(i,j,k,2) -
((1-exp(-
zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSz(i,j,k)*delta)) * (Eyz(i,j+1,k,2)+Eyx(i,j+1,k,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));
Hyz(i,j,k,3)=exp(-
(zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2) -
((1-exp(-
zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSz(i,j,k)*delta)) * (Exy(i,j+1,k,2)+Exz(i,j+1,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

Hyx(i,j,k,3)=Hyx(i,j,k,2) +((deltat/(mio0*mio(i,j,k)))/delta) * (Exy(i,j+1,k,2)+Ezx(i,j+1,k,2)-Exy(i,j,k,2)-Ezx(i,j,k,2));

end

if k>=P-PD && i>=N-PD && i>=PD+1 && i<MD-PD

Hxy(i,j,k,3)=exp(-
(zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2) -
((1-exp(-
zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSy(i,j,k)*delta)) * (Ezx(i,j+1,k,2)+Ezy(i,j+1,k,2) -
Ezx(i,j,k,2)-Ezy(i,j,k,2));
Hxz(i,j,k,3)=exp(-
(zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxz(i,j,k,2) -
((1-exp(-
zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSz(i,j,k)*delta)) * (Eyz(i,j+1,k,2)+Eyx(i,j+1,k,2)-Eyz(i,j,k,2)-Eyx(i,j,k,2));
Hyz(i,j,k,3)=exp(-
(zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2) -
((1-exp(-
zigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSz(i,j,k)*delta)) * (Exy(i,j+1,k,2)+Exz(i,j+1,k,2)-Exy(i,j,k,2)-Exz(i,j,k,2));

end

% Applying PML to 8 corners

if (k<PD+1 && i<PD+1 && j<PD+1) || (k<PD+1 && i>=M-PD && j>=PD+1) || (k<PD+1 && i<MD-1 && j>=ND-PD) || (k<PD+1 && i<MD+1 && j<ND-PD)

Hxy(i,j,k,3)=exp(-
(zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2) -
((1-exp(-
zigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))))/(zigmaSy(i,j,k)*delta)) * (Ezx(i,j+1,k,2)+Ezy(i,j+1,k,2) -
Ezx(i,j,k,2)-Ezy(i,j,k,2));

end
Hxz(i,j,k,3)=exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxz(i,j,k,2)+((1-exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSz(i,j,k)*delta))*(Ey(i,j,k+1,2)-Ez(i,j,k,2)-Ey(i,j,k,2));

Hyz(i,j,k,3)=exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2)+((1-exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSz(i,j,k)*delta))*(Ez(i,j,k+1,2)-Ex(i,j,k,2)-Ex(i,j,k,2));

Hyx(i,j,k,3)=exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyx(i,j,k,2)+((1-exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSz(i,j,k)*delta))*(Ex(i,j+1,k,2)-Ez(i,j,k,2)-Ez(i,j,k,2));

Hzx(i,j,k,3)=exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hzx(i,j,k,2)+((1-exp(-sigmaSz(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSz(i,j,k)*delta))*(Ey(i,j+1,k,2)-Ey(i,j,k,2)-Ez(i,j,k,2));

if (k>=P-PD && i<PD+1 && j<PD+1) || (k>P-PD && i>=M-PD && j<PD+1) || (k>P-PD && i<PD+1 && j>=N-PD) || (k>P-PD && i>PD+1 && j>=N-PD)

Hxy(i,j,k,3)=exp(-sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxy(i,j,k,2)+((1-exp(-sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSy(i,j,k)*delta))*(Ex(i,j+1,k,2)-Ey(i,j,k,2)-Ex(i,j,k,2));

Hxz(i,j,k,3)=exp(-sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hxz(i,j,k,2)+((1-exp(-sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSy(i,j,k)*delta))*(Ex(i,j+1,k,2)-Ez(i,j,k,2)-Ex(i,j,k,2));

Hyz(i,j,k,3)=exp(-sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k)))*Hyz(i,j,k,2)+((1-exp(-sigmaSy(i,j,k)*deltat/(mio0*mio(i,j,k))))/(sigmaSy(i,j,k)*delta))*(Ex(i,j+1,k,2)-Ey(i,j,k,2)-Ez(i,j,k,2));

end

end

%%% Applying connecting conditions or Huygens surface corrections %%
% update electric fields
% ended to M-1 & N-1 & P-1 since the outer most layer is conductive (PEC)

for i=1:1:M-1
    for j=1:1:N-1
        for k=1:1:P-1
            if (i>=PD+1 && i<M-PD) && (j>=PD+1 && j<N-PD) && (k>=PD+1 && k<P-PD)
                Ex(i,j,k,3)=Cae(i,j,k)*Ex(i,j,k,2)+(Cbe(i,j,k))*(Hz(i,j,k,3)-Hz(i,j-1,k,3)+Hy(i,j,k-1,3)-Hy(i,j,k,3));
                end
            if i~=PD+1 && k~=PD+1 && j~=PD+1
                Ey(i,j,k,3)=Cae(i,j,k)*Ey(i,j,k,2)+(Cbe(i,j,k))*(Hx(i,j,k,3)-Hx(i,j-1,k,3)+Hz(i-1,j,k,3)-Hz(i,j,k,3));
                end
            if i~=PD+1 && j~=PD+1 && k~=PD+1
                Ez(i,j,k,3)=Cae(i,j,k)*Ez(i,j,k,2)+(Cbe(i,j,k))*(Hy(i,j,k,3)-Hy(i-1,j,k,3)+Hx(i-1,j,k,3)-Hx(i,j,k,3));
                end
        end
    end
end

%%% applying FDTD for Dispersive region %%%

if (i>=i0D && i<=i1D) && (j>=j0D && j<=j1D) && (k>=k0D && k<=k1D)
    Dx(i,j,k,3)=Cae(i,j,k)*Dx(i,j,k,2)+ep0*(Cbe(i,j,k))*(Hz(i,j,k,3)-Hz(i-1,j,k,3)+Hy(i-1,j,k-1,3)-Hy(i,j,k,3));
    Ex(i,j,k,3)=(1/(ep0*(wp^2*deltat^2+2*g*deltat+4)))*((4+2*g*deltat)*Dx(i,j,k,3)-8*Dx(i,j,k,2)+(4-2*g*deltat)*Dx(i,j,k,1)+ep0*(8-2*wp^2*deltat^2)*Ex(i,j,k,3)-ep0*(wp^2*deltat^2)*Ex(i,j,k,1));
    if j~=j1D
        Dy(i,j,k,3)=Cae(i,j,k)*Dy(i,j,k,2)+ep0*(Cbe(i,j,k))*(Hx(i,j,k,3)-Hx(i-1,j,k,3)+Hz(i-1,j,k-1,3)-Hz(i,j,k,3));
        Ey(i,j,k,3)=(1/(ep0*(wp^2*deltat^2+2*g*deltat+4)))*((4+2*g*deltat)*Dy(i,j,k,3)-8*Dy(i,j,k,2)+(4-2*g*deltat)*Dy(i,j,k,1)+ep0*(8-2*wp^2*deltat^2)*Ey(i,j,k,3)-ep0*(wp^2*deltat^2)*Ey(i,j,k,1));
        end
    if k~=k1D
        Dz(i,j,k,3)=Cae(i,j,k)*Dz(i,j,k,2)+ep0*(Cbe(i,j,k))*(Hy(i,j,k,3)-Hy(i-1,j,k,3)+Hx(i-1,j,k-1,3)-Hx(i,j,k,3));
        Ez(i,j,k,3)=(1/(ep0*(wp^2*deltat^2+2*g*deltat+4)))*((4+2*g*deltat)*Dz(i,j,k,3)-8*Dz(i,j,k,2)+(4-2*g*deltat)*Dz(i,j,k,1)+ep0*(8-2*wp^2*deltat^2)*Ez(i,j,k,3)-ep0*(wp^2*deltat^2)*Ez(i,j,k,1));
        end
    end
end
2*wp^2*deltat^2)*Ez(i,j,k,2) - ep0*(wp^2*deltat^2 - 2*gp*deltat+4)*Ez(i,j,k,1));
end
end

%%% end of dispersive region %%%

%%% Applying connecting conditions or Huygens surface corrections %%%

rtest=(i j k)*[mx my mz]';
if j==j0 && (i>=i0 && i<=i1) && (k>=k0 && k<=k1)
  if i~=i1
    Ex(i,j0,k,3)=Ex(i,j0,k,3)-Cbe(i,j0-1,k)*Hrz(rtest-my,3);
  end
  if k~=k1
    Ez(i,j0,k,3)=Ez(i,j0,k,3)+Cbe(i,j0-1,k)*Hrx(rtest-my,3);
  end
end
if j==j1 && (i>=i0 && i<=i1) && (k>=k0 && k<=k1)
  if i~=i1
    Ex(i,j1,k,3)=Ex(i,j1,k,3)+Cbe(i,j1,k)*Hrz(rtest,3);
  end
  if k~=k1
    Ez(i,j1,k,3)=Ez(i,j1,k,3)-Cbe(i,j1,k)*Hrx(rtest,3);
  end
end
if k==k0 && (i>=i0 && i<=i1) && (j>=j0 && j<=j1)
  if i~=i1
    Ex(i,j,k0,3)=Ex(i,j,k0,3)+Cbe(i,j,k0-1)*Hry(rtest-mz,3);
  end
  if j~=j1
    Ey(i,j,k0,3)=Ey(i,j,k0,3)-Cbe(i,j,k0-1)*Hrx(rtest-mz,3);
  end
end
if k==k1 && (i>=i0 && i<=i1) && (j>=j0 && j<=j1)
  if i~=i1
    Ex(i,j,k1,3)=Ex(i,j,k1,3)-Cbe(i,j,k1)*Hry(rtest,3);
  end
  if j~=j1
    Ey(i,j,k1,3)=Ey(i,j,k1,3)+Cbe(i,j,k1)*Hrx(rtest,3);
  end
end
if i==i0 && (j>=j0 && j<=j1) && (k>=k0 && k<=k1)
  if j~=j1
    Ey(i0,j,k,3)=Ey(i0,j,k,3)+Cbe(i0-1,j,k)*Hrz(rtest-mx,3);
  end
  if k~=k1
    Ez(i0,j,k,3)=Ez(i0,j,k,3)-Cbe(i0-1,j,k)*Hry(rtest-mx,3);
  end
end
if i==i1 && (j>=j0 && j<=j1) && (k>=k0 && k<=k1)

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if j==j1
    Ey(i1,j,k,3)=Ey(i1,j,k,3)-Cbe(i1,j,k)*Hrz(rtest,3);
end
if k==k1
    Ez(i1,j,k,3)=Ez(i1,j,k,3)+Cbe(i1,j,k)*Hry(rtest,3);
end
end
%%% End of connecting conditions %%%

%%% End of connecting conditions %%%

%% Updating electric fields on the vacuum-PML interface surface %

if (i==PD+1 && j>PD+1 && j<N-PD && k>PD+1 && k<P-PD)
    Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hx(i,j,k,3)
- Hx(i,j,k-1,3));
    Eyx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ey(i,j,k,2)
- ((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*delta))*Hx(i,j,k,3)
- Hx(i,j,k-1,3)-Hzx(i,j,k,3));
    Ezx(i,j,k,3)=Ezx(i,j,k,2)-
    ((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)-Hz(i,j-1,k,3));
    Ezy(i,j,k,3)=Ezy(i,j,k,2)
- ((deltat/(ep0*ep(i,j,k)))/delta)*(Hzx(i,j,k,3)-Hz(i,j-1,k,3));
    Ey(i,j,k,3)=Eyx(i,j,k,3)+Eyz(i,j,k,3);
    Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
end
if i==M-PD && j>PD+1 && j<N-PD && k>P-PD
    Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hx(i,j,k,3)
+ Hx(i,j,k-1,3)-Hxy(i,j,k-1,3));
    Eyx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ey(i,j,k,2)
- ((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*delta))*Hx(i,j,k,3)
+ Hx(i,j,k-1,3)-Hzx(i,j,k,3));
    Ezx(i,j,k,3)=Ezx(i,j,k,2)-
    ((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)+Hz(i,j-1,k,3)-Hzx(i,j-1,k,3));
    Ezy(i,j,k,3)=Ezy(i,j,k,2)
+ ((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)-Hzx(i,j-1,k,3));
    Eyi(i,j,k,3)=Ey(i,j,k,3)+Ezy(i,j,k,3);
    Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
end
if j==PD+1 && i>PD+1 && i<MD-PD && k>P-PD
    Exy(i,j,k,3)=exp(-
    (zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)
+ Hz(i,j-1,k,3)-Hzx(i,j-1,k,3));
    Exz(i,j,k,3)=Exz(i,j,k,2)-
    ((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)-Hzx(i,j-1,k,3));
    Ezy(i,j,k,3)=Ezy(i,j,k,2)
+ ((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)-Hzx(i,j-1,k,3));
    Eyi(i,j,k,3)=Ey(i,j,k,3)+Ezy(i,j,k,3);
    Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
end

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Ezx(i,j,k,3)=Ezx(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hy(i,j,k,3)-Hy(i-1,j,k,3));
Ezy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)-((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Ha(i,j,k,3)-Ha(i,j-1,k,3));
Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
end

if j==N && i>PD+1 && k>PD+1 && k<P-PD
Exy(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hx(i,j,k,3)+Hx(i-1,j,k,3));
Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Ey(i,j,k,3)-Ey(i,j,k-1,3));
Eyz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hx(i,j,k,3)+Hx(i,j,k-1,3));
Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
end

if k==P && i>PD+1 && j<PD+1 && j>N-PD
Exy(i,j,k,3)=Exy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)-Hz(i,j-1,k,3));
Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hy(i,j,k,3)+Hy(i,j,k-1,3));
Eyz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hx(i,j,k,3)+Hx(i,j,k-1,3));
Ey(i,j,k,3)=Eyz(i,j,k,3)+Exz(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Exy(i,j,k,3);
end

if k==P-PD && i>PD+1 && j<N-PD
Exy(i,j,k,3)=Exy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)-Hz(i,j-1,k,3));
Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hy(i,j,k,3)+Hy(i,j,k-1,3));
Eyz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hx(i,j,k,3)+Hx(i,j,k-1,3));
Ey(i,j,k,3)=Eyz(i,j,k,3)+Exz(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Exy(i,j,k,3);
End


zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*deltat)) *(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hx(i,j,k-1,3));

Ex(i,j,k,3)=Ey(i,j,k,2)-((deltat/(ep0*ep(i,j,k)))/delta) *(Hx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i-1,j,k,3)-Hzy(i-1,j,k,3));

Ex(i,j,k,3)=Ey(i,j,k,3)+Exz(i,j,k,3);
Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
end

%%%%% Line edges of vacuum-PML interface

%%%%% bottom edges
if k==PD+1 && i==PD+1 && j==PD+1 && j<N-PD
Exy(i,j,k,3)=Ey(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta) *(Hz(i,j,k,3)-Hz(i,j-1,k,3));

Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-(1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat)) *(Hy(i,j,k,3)-Hxy(i,j,k-1,3)-Hy(i,j-1,k,3));

Ey(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ey(i,j,k,2)+((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat)) *(Hz(i,j,k,3)-Hzx(i,j,k-1,3)-Hz(i,j-1,k,3));
if j~PD+1
Exy(i,j,k,3)=Eyz(i,j,k,3)+Ey(i,j,k,3);
Ey(i,j,k,3)=Ezx(i,j,k,3)+Ex(i,j,k,3);
end

if k==PD+1 && i==PD+1 && j=PD+1 && i<N-M-PD
Exy(i,j,k,3)=Eyz(i,j,k,3)+Exz(i,j,k,3);
Ey(i,j,k,3)=Ezx(i,j,k,3)+Ex(i,j,k,3);
end

if k==PD+1 && j==PD+1 && i>=PD+1 && i<M-PD
Exy(i,j,k,3)=Ey(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta) *(Hz(i,j,k,3)-Hz(i,j-1,k,3));

Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-(1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat)) *(Hy(i,j,k,3)-Hxy(i,j,k-1,3)-Hy(i,j-1,k,3));

Ey(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ey(i,j,k,2)+((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat)) *(Hz(i,j,k,3)-Hzx(i,j,k-1,3)-Hz(i,j-1,k,3));
if i~PD+1
Ey(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ey(i,j,k,2)+((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat)) *(Hz(i,j,k,3)-Hzx(i,j,k-1,3)-Hz(i,j-1,k,3));
Eyx(i,j,k,3)=Eyx(i,j,k,2)-((deltat/(ep0*ep(i,j,k)))/delta)*(Hz(i,j,k,3)-Hz(i-1,j,k,3));
Ezx(i,j,k,3)=Ezx(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hy(i,j,k,3)-Hy(i-1,j,k,3));
Ezy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)-((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmay(i,j,k)*delta)))*(Hx(i,j,k,3)-Hxy(i,j-1,k,3)-Hxz(i,j-1,k,3));
end
Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
End

if k==PD+1 && i==M-PD && j>=PD+1 && j<N-PD
Eyx(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2)-((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmay(i,j,k)*delta)))*(Hz(i,j,k,3)-Hz(i-1,j,k,3));
Ezy(i,j,k,3)=Ezy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j-1,k,3)-Hxz(i,j-1,k,3));
end
Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
End

if k==PD+1 && j==N-PD && i>=PD+1 && i<M-PD
Ezx(i,j,k,3)=Ezx(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
Ezy(i,j,k,3)=Ezy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j-1,k,3)-Hxz(i,j-1,k,3));
end
Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
End

if k==PD+1 && j==M-PD && i>=PD+1 && i<N-PD
Ezy(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta)))*(Hx(i,j,k,3)+Hyz(i,j,k,3)-Hx(i,j-1,k,3)-Hyz(i,j-1,k,3));
Ezx(i,j,k,3)=Ezx(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
end
Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
End

if k==PD+1 && j==N-PD && i>=PD+1 && i<M-PD
Ezx(i,j,k,3)=Ezx(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
Ezy(i,j,k,3)=Ezy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j-1,k,3)-Hxz(i,j-1,k,3));
end
Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
Ez(i,j,k,3)=Ezx(i,j,k,3)+Ezy(i,j,k,3);
End
% 4 side edges

if k~=PD+1
    Exy(i,j,k,3)=exp(-
        zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))
            *Exy(i,j,k,2)+((1-exp(-
        zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/
        (zigmay(i,j,k)*delta))
            *(Hz(i,j,k,3)-Hzx(i,j-1,k,3)
                -Hzy(i,j-1,k,3));
    Exz(i,j,k,3)=Exz(i,j,k,2)-
        ((deltat/(ep0*ep(i,j,k)))/delta)
            *(Hy(i,j,k,3)-Hyx(i,j-1,k,3));
    Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*
        (Hx(i,j,k,3)-Hzx(i,j,k-1,3));
    Eyx(i,j,k,3)=exp(-
        zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))
            *Eyx(i,j,k,2)-
        ((1-exp(-
        zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/
        (zigmay(i,j,k)*delta));
end

if i==M-PD && j==PD+1 && k>=PD+1 && k<P-PD
    Ezx(i,j,k,3)=exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))
            *Ezx(i,j,k,2)+((1-exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/
        (zigmax(i,j,k)*delta))
            *(Hx(i,j,k,3)-Hyz(i,j-1,k,3)
                -Hzx(i,j-1,k,3));
    Ezy(i,j,k,3)=exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))
            *Ezy(i,j,k,2)-
        ((1-exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/
        (zigmax(i,j,k)*delta));
    Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
end

if k~=PD+1
    Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*
        (Hx(i,j,k,3)-Hxz(i,j,k-1,3));
    Eyx(i,j,k,3)=exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))
            *Eyx(i,j,k,2)-
        ((1-exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/
        (zigmax(i,j,k)*delta));
end

if i==M-PD && j==N-PD && k>=PD+1 && k<P-PD
    Ezx(i,j,k,3)=exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))
            *Ezx(i,j,k,2)+((1-exp(-
        zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/
        (zigmax(i,j,k)*delta));
end
\[
\text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) / (\text{zigmax}(i,j,k) \cdot \text{delta})) \cdot (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hy}(i-1,j,k,3));
\]
\[
\text{Ezy}(i,j,k,3) = \text{exp}(- \text{zigmay}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Ezy}(i,j,k,2) - ((1 - \text{exp}(- \text{zigmay}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)))) / (\text{zigmay}(i,j,k) \cdot \text{delta})) \cdot (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3));
\]
\[
\text{Ez}(i,j,k,3) = \text{Ezx}(i,j,k,3) + \text{Ezy}(i,j,k,3);
\]
\end{verbatim}

```
if i==PD+1 && j==N-PD && k>=PD+1 && k<P-PD
  if k~=PD+1
    \text{Exy}(i,j,k,3) = \text{exp}(- \text{zigmay}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Exy}(i,j,k,2) + ((1 - \text{exp}(- \text{zigmay}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)))) / (\text{zigmay}(i,j,k) \cdot \text{delta})) \cdot (\text{Hzx}(i,j,k,3) + \text{Hz}(i,j-1,k,3));
    \text{Exz}(i,j,k,3) = \text{exp}(- \text{zigmaz}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Exz}(i,j,k,2) - ((1 - \text{exp}(- \text{zigmaz}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)))) / (\text{zigmaz}(i,j,k) \cdot \text{delta})) \cdot (\text{Hyz}(i,j,k,3) + \text{Hy}(i,j,k-1,3));
  end
  \text{Ezx}(i,j,k,3) = \text{exp}(- \text{zigmaz}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Ezx}(i,j,k,2) + ((1 - \text{exp}(- \text{zigmaz}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)))) / (\text{zigmaz}(i,j,k) \cdot \text{delta})) \cdot (\text{Hyz}(i,j,k,3) + \text{Hy}(i,j,k-1,3));
  \text{Ezy}(i,j,k,3) = \text{exp}(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Ezy}(i,j,k,2) - ((1 - \text{exp}(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)))) / (\text{zigmax}(i,j,k) \cdot \text{delta})) \cdot (\text{Hzx}(i,j,k,3) + \text{Hz}(i,j,k-1,3));
  \text{Ex}(i,j,k,3) = \text{Exy}(i,j,k,3) + \text{Exz}(i,j,k,3);
  \text{Ey}(i,j,k,3) = \text{Ezx}(i,j,k,3) + \text{Ezy}(i,j,k,3);
end
```

```
\begin{verbatim}
%%%% Top edges %%%%%
if k==P-PD && i==PD+1 && j==PD+1 && j<N-PD
  if j~PD+1
    \text{Exy}(i,j,k,3) = \text{exp}(- \text{zigmay}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Exy}(i,j,k,2) + ((1 - \text{exp}(- \text{zigmay}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)))) / (\text{zigmay}(i,j,k) \cdot \text{delta})) \cdot (\text{Hzx}(i,j,k,3) + \text{Hz}(i,j-1,k,3));
    \text{Exz}(i,j,k,3) = \text{exp}(- \text{zigmaz}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Exz}(i,j,k,2) - ((1 - \text{exp}(- \text{zigmaz}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)))) / (\text{zigmaz}(i,j,k) \cdot \text{delta})) \cdot (\text{Hyz}(i,j,k,3) + \text{Hy}(i,j,k-1,3));
  end
  \text{Ex}(i,j,k,3) = \text{Exy}(i,j,k,3) + \text{Exz}(i,j,k,3);
  \text{Ey}(i,j,k,3) = \text{Ezx}(i,j,k,3) + \text{Ezy}(i,j,k,3);
end
```

```
if k==P-PD && i==PD+1 && j==PD+1 && j<N-PD
```
\begin{verbatim}
Exy(i,j,k,3)=exp(-
  zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-exp(-
  zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))
  *(Hz(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3)));
Exz(i,j,k,3)=exp(-
  zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)+((1-
  exp(-
  zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))
  *(Hyx(i,j,k,3)+Hyz(i,j,k,3)-Hxy(i,j,k-1,3));

if i==PD+1
  Eyz(i,j,k,3)=exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2)+((1-
    exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))
    *(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
  Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
end
if k==P-PD && i==M-PD && j>=PD+1 && j<N-PD
  Eyz(i,j,k,3)=exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2)+((1-
    exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))
    *(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
  Ey(i,j,k,3)=Eyz(i,j,k,3)+Eyx(i,j,k,3);
end
if k==P-PD && j==N-PD && i>=PD+1 && i<M-PD
  Exy(i,j,k,3)=exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-
    exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))
    *(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
  Ex(i,j,k,3)=Exy(i,j,k,3)+Exz(i,j,k,3);
end
if j==1 && k==1 && j==PD+1 && k==PD+1
  Exy(i,j,k,3)=Exy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/
    delta)*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
  Exz(i,j,k,3)=Exz(i,j,k,2)-
    ((deltat/(ep0*ep(i,j,k)))/
    delta)*(Hyx(i,j,k,3)+Hyz(i,j,k,3)-Hxy(i,j,k-1,3)-Hyx(i,j,k-1,3));
end

% Applying PML Conditions inside 6 side layers

if i<PD+1 && j==PD+1 && j<N-PD && k==PD+1 && k<P-PD
  if j==1 && k==1 && j==PD+1 && k==PD+1
    Exy(i,j,k,3)=Exy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/
      delta)*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
    Exz(i,j,k,3)=Exz(i,j,k,2)-
      ((deltat/(ep0*ep(i,j,k)))/
      delta)*(Hyx(i,j,k,3)+Hyz(i,j,k,3)-Hxy(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end

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\end{verbatim}
if i~1 && k~1 && k~PD+1
Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
Eyx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2)-(((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))))^(zigmax(i,j,k)*delta))*Hzx(i,j,k,3)-Hzx(i-1,j,k,3)-Hzx(i-1,1,k,3))-Eyz(i,j,k,3);
end
if i~1 && j~1 && j~PD+1
Ezx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))))^(zigmax(i,j,k)*delta))*Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
Ezy(i,j,k,3)=Ezy(i,j,k,2)-((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
end
end
if i~M-PD && j~PD+1 && j<N-PD && k~PD+1 && k<P-PD
if j~PD+1 && k~PD+1
Exy(i,j,k,3)=Exy(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
Exz(i,j,k,3)=Exz(i,j,k,2)-((deltat/(ep0*ep(i,j,k)))/delta)*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end
if i~M-PD && k~PD+1
Ezx(i,j,k,3)=Exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))))^(zigmax(i,j,k)*delta))*Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
Ezy(i,j,k,3)=Ezy(i,j,k,2)-((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
end
end
if j<PD+1 && i~PD+1 && i<M-PD && k~PD+1 && k<P-PD
if j~1 && k~1 && k~PD+1
Exy(i,j,k,3)=Exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))))^(zigmay(i,j,k)*delta))*Hzx(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end
end
\[
\begin{align*}
Exz(i,j,k,3) &= Exz(i,j,k,2) - \\
&\quad \left( \frac{\text{deltat}}{\text{ep0} \times \text{ep}(i,j,k)} \right) \times \left( Hyz(i,j,k,3) + Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hyx(i,j,k-1,3) \right); \\
&\quad \text{end if}
\end{align*}
\]

\[
\begin{align*}
Eyz(i,j,k,3) &= Eyz(i,j,k,2) + \left( \frac{\text{deltat}}{\text{ep0} \times \text{ep}(i,j,k)} \right) \times \left( Hxy(i,j,k,3) + Hxz(i,j,k,3) - Hxy(i,j,k-1,3) - Hxz(i,j,k-1,3) \right); \\
&\quad \text{end if} \\
Ey(i,j,k,3) &= \left( \frac{\text{deltat}}{\text{ep0} \times \text{ep}(i,j,k)} \right) \times \left( Hxy(i,j,k,3) + Hzy(i,j,k,3) - Hxy(i-1,j,k,3) - Hzy(i-1,j,k,3) \right); \\
&\quad \text{end if}
\end{align*}
\]

\[
\begin{align*}
Exy(i,j,k,3) &= \exp(-\text{zigmay}(i,j,k) \times \text{deltat} / \text{ep0} \times \text{ep}(i,j,k)) \times Ey(i,j,k,2) - \\
&\quad \left( 1 - \exp(-\text{zigmay}(i,j,k) \times \text{deltat} / \text{ep0} \times \text{ep}(i,j,k)) \right) \times \left( Hzy(i,j,k,3) + Hzx(i,j,k,3) - Hzy(i,j-1,k,3) - Hzx(i,j-1,k,3) \right); \\
&\quad \text{end if} \\
Exz(i,j,k,3) &= Exz(i,j,k,2) - \left( \frac{\text{deltat}}{\text{ep0} \times \text{ep}(i,j,k)} \right) \times \left( Hyz(i,j,k,3) + Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hyx(i,j,k-1,3) \right); \\
&\quad \text{end if}
\end{align*}
\]

\[
\begin{align*}
Ey(i,j,k,3) &= \exp(-\text{zigmay}(i,j,k) \times \text{deltat} / \text{ep0} \times \text{ep}(i,j,k)) \times Ey(i,j,k,2) + \\
&\quad \left( 1 - \exp(-\text{zigmay}(i,j,k) \times \text{deltat} / \text{ep0} \times \text{ep}(i,j,k)) \right) \times \left( Hxy(i,j,k,3) + Hxz(i,j,k,3) - Hxy(i,j-1,k,3) - Hxz(i,j-1,k,3) \right); \\
&\quad \text{end if}
\end{align*}
\]

\[
\begin{align*}
Eyz(i,j,k,3) &= Eyz(i,j,k,2) + \left( \frac{\text{deltat}}{\text{ep0} \times \text{ep}(i,j,k)} \right) \times \left( Hxz(i,j,k,3) + Hzy(i,j,k,3) - Hxz(i,j-1,k,3) - Hzy(i,j-1,k,3) \right); \\
&\quad \text{end if} \\
Eyx(i,j,k,3) &= Eyx(i,j,k,2) - \left( \frac{\text{deltat}}{\text{ep0} \times \text{ep}(i,j,k)} \right) \times \left( Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3) \right); \\
&\quad \text{end if}
\end{align*}
\]

\[
\begin{align*}
Ezy(i,j,k,3) &= \exp(-\text{zigmay}(i,j,k) \times \text{deltat} / \text{ep0} \times \text{ep}(i,j,k)) \times Ezy(i,j,k,2) - \\
&\quad \left( 1 - \exp(-\text{zigmay}(i,j,k) \times \text{deltat} / \text{ep0} \times \text{ep}(i,j,k)) \right) \times \left( Hxy(i,j,k,3) + Hxz(i,j,k,3) - Hxy(i,j-1,k,3) - Hxz(i,j-1,k,3) \right); \\
&\quad \text{end if}
\end{align*}
\]

\[
\begin{align*}
Exx(i,j,k,3) &= Exx(i,j,k,2) - \left( \frac{\text{deltat}}{\text{ep0} \times \text{ep}(i,j,k)} \right) \times \left( Hyz(i,j,k,3) + Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hyx(i,j,k-1,3) \right); \\
&\quad \text{end if}
\end{align*}
\]
\[
\begin{align*}
\text{Exy}(i,j,k,3) &= \text{Exy}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hzx}(i,j,k,3) + \text{Hzy}(i,j,k,3) - \text{Hzx}(i,j-1,k,3) - \text{Hzy}(i,j-1,k,3)) \\
\text{Exz}(i,j,k,3) &= \exp\left(-\text{zigmaz}(i,j,k) \times \frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))}\right) \times \text{Exz}(i,j,k,2) - \left(\frac{1 - \exp\left(-\text{zigmaz}(i,j,k) \times \frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))}\right)}{\text{zigmaz}(i,j,k) \times \text{delta}}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j,k-1,3) - \text{Hyx}(i,j,k-1,3)) \\
\text{Eyz}(i,j,k,3) &= \exp\left(-\text{zigmaz}(i,j,k) \times \frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))}\right) \times \text{Eyz}(i,j,k,2) + \left(\frac{1 - \exp\left(-\text{zigmaz}(i,j,k) \times \frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))}\right)}{\text{zigmaz}(i,j,k) \times \text{delta}}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j,k-1,3) - \text{Hxz}(i,j,k-1,3)) \\
\text{Eyx}(i,j,k,3) &= \text{Eyx}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hzx}(i,j,k,3) + \text{Hzy}(i,j,k,3) - \text{Hzx}(i,j-1,k,3) - \text{Hzy}(i,j-1,k,3)) \\
\text{Exz}(i,j,k,3) &= \text{Exz}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j-1,k,3) - \text{Hyx}(i,j-1,k,3)) \\
\text{Ezy}(i,j,k,3) &= \text{Ezy}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3)) \\
\text{Ezx}(i,j,k,3) &= \text{Ezx}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j-1,k,3) - \text{Hyx}(i,j-1,k,3)) \\
\text{Ezy}(i,j,k,3) &= \text{Ezy}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3)) \\
\text{Ezx}(i,j,k,3) &= \text{Ezx}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j-1,k,3) - \text{Hyx}(i,j-1,k,3)) \\
\text{Ezy}(i,j,k,3) &= \text{Ezy}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3)) \\
\text{Ezx}(i,j,k,3) &= \text{Ezx}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j-1,k,3) - \text{Hyx}(i,j-1,k,3)) \\
\text{Ezy}(i,j,k,3) &= \text{Ezy}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3)) \\
\text{Ezx}(i,j,k,3) &= \text{Ezx}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j-1,k,3) - \text{Hyx}(i,j-1,k,3)) \\
\text{Ezy}(i,j,k,3) &= \text{Ezy}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3)) \\
\text{Ezx}(i,j,k,3) &= \text{Ezx}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j-1,k,3) - \text{Hyx}(i,j-1,k,3)) \\
\text{Ezy}(i,j,k,3) &= \text{Ezy}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3)) \\
\text{Ezx}(i,j,k,3) &= \text{Ezx}(i,j,k,2) + \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3) - \text{Hyz}(i,j-1,k,3) - \text{Hyx}(i,j-1,k,3)) \\
\text{Ezy}(i,j,k,3) &= \text{Ezy}(i,j,k,2) - \left(\frac{\text{deltat}}{(\text{ep0} \cdot \text{ep}(i,j,k))} / \text{delta}\right) \times (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3) - \text{Hxy}(i,j-1,k,3) - \text{Hxz}(i,j-1,k,3)) \\
\end{align*}
\]

%% Applying PML conditions to 12 edges
% 4 bottom edges %%%
if $k<=PD+1$ && $i<=PD+1$ && $j>=PD+1$ && $j<N-PD$
if $k=1$ && $i=PD+1$ && $j=PD+1$
$E_{xy}(i,j,k,3)=Exy(i,j,k,2)+((\text{deltat}/(ep0*ep(i,j,k)))/\delta)*(Hzx(i,j,k,3)+Hzy(i,j-1,k,3)-Hyz(i,j-1,k,3))$;
$Exz(i,j,k,3)=\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k))))/(\text{zigmaz}(i,j,k)*\delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hyz(i,j-1,k,3));
end
if $i=1$ && $k=1$ && $(i<=PD+1 && k<=PD+1)$
$E_{yz}(i,j,k,3)=\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hyz(i,j-1,k,3));
$E_{yx}(i,j,k,3)=\exp(-\text{zigmax}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hyz(i,j-1,k,3));
end
if $i=1$ && $k=1$ && $(i<PD+1 && k==PD+1)$
$E_{yz}(i,j,k,3)=\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i-1,j,k,3)-Hyz(i-1,j,k,3));
$E_{yx}(i,j,k,3)=\exp(-\text{zigmax}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i-1,j,k,3)-Hyz(i-1,j,k,3));
end
if $i=1$ && $k=-1$ && $(i<=PD+1 && k>PD+1)$
$E_{zx}(i,j,k,3)=\exp(-\text{zigmax}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hyz(i,j-1,k,3));
$E_{zy}(i,j,k,3)=E_{zy}(i,j,k,2)-((\text{deltat}/(ep0*ep(i,j,k)))/\delta)*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i-1,j,k,3)-Hyz(i-1,j,k,3));
end
if $j=1$ && $k=1$ && $(j<=PD+1 && j>PD+1)$
$Exy(i,j,k,3)=Exy(i,j,k,2)+((\text{deltat}/(ep0*ep(i,j,k)))/\delta)*(Hzx(i,j,k,3)+Hzy(i,j-1,k,3)-Hyz(i,j-1,k,3))$;
$Exz(i,j,k,3)=\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k))))/(\text{zigmaz}(i,j,k)*\delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hyz(i,j-1,k,3));
end
if $j=1$ && $k=-1$ && $(j<=PD+1 && j<PD+1)$
$Exy(i,j,k,3)=Exy(i,j,k,2)+((\text{deltat}/(ep0*ep(i,j,k)))/\delta)*(Hzx(i,j,k,3)+Hzy(i,j-1,k,3)-Hyz(i,j-1,k,3))$;
$Exz(i,j,k,3)=\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k))))/(\text{zigmaz}(i,j,k)*\delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hyz(i,j-1,k,3));
end
if $j=1$ && $k=1$ && $(j<=PD+1 && j>PD+1)$
$Exy(i,j,k,3)=Exy(i,j,k,2)+((\text{deltat}/(ep0*ep(i,j,k)))/\delta)*(Hzx(i,j,k,3)+Hzy(i,j-1,k,3)-Hyz(i,j-1,k,3))$;
$Exz(i,j,k,3)=\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(ep0*ep(i,j,k))))/(\text{zigmaz}(i,j,k)*\delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hyz(i,j-1,k,3));
end
if $j=1$ && $k=-1$ && $(j<=PD+1 && j<PD+1)$
$Exy(i,j,k,3)=Exy(i,j,k,2)+((\text{deltat}/(ep0*ep(i,j,k)))/\delta)*(Hzx(i,j,k,3)+Hzy(i,j-1,k,3)-Hyz(i,j-1,k,3))$;
\[
\text{Exz}(i,j,k,3) = \exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))) \times \text{Exz}(i,j,k,2) - ((1-\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))))/(\text{zigmaz}(i,j,k)*\text{delta})) \times (\text{Hyz}(i,j,k,3)+\text{Hyx}(i,j,k,3)-\text{Hyz}(i,j,k-1,3)-\text{Hyx}(i,j,k-1,3));
\]

if \(k=1 \&\& i=\text{PD+1} \&\& j=\text{PD+1}\)

\[
\text{Eyz}(i,j,k,3) = \exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))) \times \text{Eyz}(i,j,k,2) + ((1-\exp(-\text{zigmaz}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))))/(\text{zigmaz}(i,j,k)*\text{delta})) \times (\text{Hxy}(i,j,k,3)+\text{Hxz}(i,j,k,3)-\text{Hxy}(i,j,k-1,3)-\text{Hxz}(i,j,k-1,3)) - ((\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k)))/\text{delta}) \times (\text{Hzx}(i,j,k,3)+\text{Hzy}(i,j,k,3)-\text{Hzx}(i,j,k-1,3)-\text{Hzy}(i,j,k-1,3));
\]

end

if \(j=1 \&\& i=\text{PD+1} \&\& k=\text{PD+1}\)

\[
\text{Exy}(i,j,k,3) = \exp(-\text{zigmay}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))) \times \text{Exy}(i,j,k,2) - ((1-\exp(-\text{zigmay}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))))/(\text{zigmay}(i,j,k)*\text{delta})) \times (\text{Hzx}(i,j,k,3)+\text{Hzy}(i,j,k,3)-\text{Hzx}(i,j,k-1,3)-\text{Hzy}(i,j,k-1,3)) - ((\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k)))/\text{delta}) \times (\text{Hzx}(i,j,k,3)+\text{Hzy}(i,j,k,3)-\text{Hzx}(i,j,k-1,3)-\text{Hzy}(i,j,k-1,3));
\]

end

if \(k=1 \&\& (i=\text{M-PD} \&\& k=\text{PD+1})\)

\[
\text{Ezx}(i,j,k,3) = \exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))) \times \text{Ezx}(i,j,k,2) + ((1-\exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k)*\text{delta})) \times (\text{Hzx}(i,j,k,3)+\text{Hzy}(i,j,k,3)-\text{Hzx}(i,j,k-1,3)-\text{Hzy}(i,j,k-1,3));
\]

end

if \(j=\text{PD+1} \&\& k=\text{PD+1}\)

\[
\text{Ezx}(i,j,k,3) = \exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))) \times \text{Ezx}(i,j,k,2) + ((1-\exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k)*\text{delta})) \times (\text{Hzx}(i,j,k,3)+\text{Hzy}(i,j,k,3)-\text{Hzx}(i,j,k-1,3)-\text{Hzy}(i,j,k-1,3));
\]

end

if \(k=1 \&\& (i=\text{PD+1} \&\& k=\text{PD+1})\)

\[
\text{Ezx}(i,j,k,3) = \exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))) \times \text{Ezx}(i,j,k,2) + ((1-\exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k)*\text{delta})) \times (\text{Hzx}(i,j,k,3)+\text{Hzy}(i,j,k,3)-\text{Hzx}(i,j,k-1,3)-\text{Hzy}(i,j,k-1,3));
\]

end

if \(j=\text{PD+1} \&\& k=\text{PD+1}\)

\[
\text{Ezx}(i,j,k,3) = \exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))) \times \text{Ezx}(i,j,k,2) + ((1-\exp(-\text{zigmax}(i,j,k)*\text{deltat}/(\text{ep0} \times \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k)*\text{delta})) \times (\text{Hzx}(i,j,k,3)+\text{Hzy}(i,j,k,3)-\text{Hzx}(i,j,k-1,3)-\text{Hzy}(i,j,k-1,3));
\]

end
Ezy(i,j,k,3)=Ezy(i,j,k,2)-
((deltat/(ep0*ep(i,j,k)))/delta) *(Hxy(i,j,k,3)+Hzx(i,j,k,3)-Hxy(i,j-1,k,3)-Hzx(i,j-1,k,3));
end
end
if k<=PD+1 && j>=N-PD && i>=PD+1 && i<M-PD
  if k-1 && (k<=PD+1 && j>N-PD)
    Exy(i,j,k,3)=exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))) *Exy(i,j,k,2)+((1-
    exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
  end
if k-1 && (k<PD+1 && j==N-PD)
    Exy(i,j,k,3)=exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))) *Exy(i,j,k,2)-((1-
    exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
end
if k-1 && i-1-PD+1
  Eyz(i,j,k,3)=exp(-
  zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))) *Eyz(i,j,k,2)+((1-
  exp(-
  zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hxy(i,j,k,3)+Hyx(i,j,k,3)-Hxy(i,j-1,k,3)-Hyx(i,j-1,k,3));
end
if i-1 && k-1-PD+1
  Exz(i,j,k,3)=Exz(i,j,k,2)-
  ((deltat/(ep0*ep(i,j,k)))/delta) *(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
end
if i-1 && k-1-PD+1 && j-1-PD+1
  Ezx(i,j,k,3)=Ezx(i,j,k,2)+
  ((deltat/(ep0*ep(i,j,k)))/delta) *(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
end

%%% 4 side edges
if i<=PD+1 && j<=PD+1 && k<=PD+1 && k<P-PD
  if j-1 && k-1-PD+1 && i-1-PD+1
    Exy(i,j,k,3)=exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))) *Exy(i,j,k,2)+((1-
    exp(-
    zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
  end
if j-1 && k-1-PD+1 && j-1-PD+1
  Exz(i,j,k,3)=Exz(i,j,k,2)-
  ((deltat/(ep0*ep(i,j,k)))/delta) *(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
end
if i-1 && k-1-PD+1 && j-1-PD+1
Eyz(i, j, k, 3) = Eyz(i, j, k, 2) + \left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * (Hxy(i, j, k, 3) + Hxz(i, j, k, 3) - Hxy(i, j, k-1, 3) - Hxz(i, j, k-1, 3));

Ey(i, j, k, 3) = \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k))) * Ey(i, j, k, 2) - \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hxy(i, j, k, 3) + Hyz(i, j, k, 3) - Hxy(i-1, j, k, 3) - Hyz(i-1, j, k, 3));

end

if i ~= 1 && j ~= 1 && (i <= PD + 1 && j < PD + 1)

Ezx(i, j, k, 3) = \exp(-
\left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * Ezx(i, j, k, 2) + \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hyz(i, j, k, 3) + Hyx(i, j, k, 3) - Hyz(i-1, j, k, 3) - Hyx(i-1, j, k, 3));

Ezy(i, j, k, 3) = \exp(-
\left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * Ezy(i, j, k, 2) - \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hxy(i, j, k, 3) + Hxz(i, j, k, 3) - Hxy(i-1, j, k, 3) - Hxz(i-1, j, k, 3));

end

if i >= M - PD && j <= PD + 1 && k > PD + 1 && k < PD + 1

Ezx(i, j, k, 3) = \exp(-
\left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * Ezx(i, j, k, 2) + \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hyz(i, j, k, 3) + Hyx(i, j, k, 3) - Hyz(i-1, j, k, 3) - Hyx(i-1, j, k, 3));

Ezy(i, j, k, 3) = \exp(-
\left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * Ezy(i, j, k, 2) - \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hxy(i, j, k, 3) + Hxz(i, j, k, 3) - Hxy(i-1, j, k, 3) - Hxz(i-1, j, k, 3));

end

if k = PD + 1 && j = PD + 1

Eyz(i, j, k, 3) = Eyz(i, j, k, 2) + \left( \frac{\text{deltat}}{(ep0 * ep(i, j, k))} \right) * (Hxy(i, j, k, 3) + Hxz(i, j, k, 3) - Hxy(i, j, k-1, 3) - Hxz(i, j, k-1, 3));

Eyx(i, j, k, 3) = \exp(-
\left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * Eyx(i, j, k, 2) - \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hxz(i, j, k, 3) + Hyz(i, j, k, 3) - Hxz(i-1, j, k, 3) - Hyz(i-1, j, k, 3));

end

if j = PD + 1 && i > M - PD

Ezx(i, j, k, 3) = \exp(-
\left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * Ezx(i, j, k, 2) + \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hyz(i, j, k, 3) + Hyx(i, j, k, 3) - Hyz(i-1, j, k, 3) - Hyx(i-1, j, k, 3));

Ezy(i, j, k, 3) = \exp(-
\left( \frac{\text{deltat}}{(\varepsilon_0 \varepsilon(i, j, k))} \right) * Ezy(i, j, k, 2) - \left( (1 - \exp(-zigmax(i, j, k) * \text{deltat} / (ep0 * ep(i, j, k)))) / \text{zigmax}(i, j, k) * delta \right) * (Hxy(i, j, k, 3) + Hxz(i, j, k, 3) - Hxy(i-1, j, k, 3) - Hxz(i-1, j, k, 3));

end

end
if \( j = 1 \) \&\& (\( j \neq PD + 1 \) \&\& \( i = M - PD \))

\[
E_{zx}(i,j,k,3) = \exp(-zigmax(i,j,k) \cdot deltat/(ep0 \cdot ep(i,j,k))) \cdot E_{zx}(i,j,k,2) + ((1 - \exp(-zigmax(i,j,k) \cdot deltat)/(ep0 \cdot ep(i,j,k))))/(zigmax(i,j,k) \cdot delta) \cdot (Hyz(i,j,k,3) + Hyx(i,j,k,3) - Hyz(i-1,j,k,3) - Hyx(i-1,j,k,3));
\]

end

end

if \( i > M - PD \) \&\& \( j > N - PD \) \&\& \( k > PD + 1 \) \&\& \( k < P - PD \)

end

if \( k = PD + 1 \)

E_{xy}(i,j,k,3) = E_{xy}(i,j,k,2) + ((deltat/(ep0 \cdot ep(i,j,k))) \cdot (Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3));
\]

end

if \( i > M - PD \) \&\& \( j = N - PD \)

E_{zx}(i,j,k,3) = \exp(-zigmax(i,j,k) \cdot deltat/(ep0 \cdot ep(i,j,k))) \cdot E_{zx}(i,j,k,2) + ((1 - \exp(-zigmax(i,j,k) \cdot deltat)/(ep0 \cdot ep(i,j,k))))/(zigmax(i,j,k) \cdot delta) \cdot (Hyz(i,j,k,3) + Hyx(i,j,k,3) - Hyz(i,j-1,k,3) - Hyx(i,j-1,k,3));
\]

end

if \( i > M - PD \) \&\& \( j = N - PD \)

E_{yx}(i,j,k,3) = \exp(-zigmay(i,j,k) \cdot deltat/(ep0 \cdot ep(i,j,k))) \cdot E_{yx}(i,j,k,2) + ((1 - \exp(-zigmay(i,j,k) \cdot deltat)/(ep0 \cdot ep(i,j,k))))/(zigmay(i,j,k) \cdot delta) \cdot (Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3));
\]

end

if \( i = PD + 1 \) \&\& \( j > N - PD \)

E_{xy}(i,j,k,3) = \exp(- zigmax(i,j,k) \cdot deltat/(ep0 \cdot ep(i,j,k))) \cdot E_{xy}(i,j,k,2) + ((1 - \exp(- zigmax(i,j,k) \cdot deltat)/(ep0 \cdot ep(i,j,k))))/(zigmax(i,j,k) \cdot delta) \cdot (Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3));
\]

end

if \( i = PD + 1 \) \&\& \( j = N - PD \)

end

if \( i = PD + 1 \) \&\& \( j = N - PD \)

E_{yx}(i,j,k,3) = \exp(- zigmay(i,j,k) \cdot deltat/(ep0 \cdot ep(i,j,k))) \cdot E_{yx}(i,j,k,2) + ((1 - \exp(- zigmay(i,j,k) \cdot deltat)/(ep0 \cdot ep(i,j,k))))/(zigmay(i,j,k) \cdot delta) \cdot (Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3));
\]

end

if \( k = PD + 1 \)

E_{xy}(i,j,k,3) = E_{xy}(i,j,k,2) + ((deltat/(ep0 \cdot ep(i,j,k))) \cdot (Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3));
\]

end

if \( k = PD + 1 \)

E_{yx}(i,j,k,3) = \exp(- zigmax(i,j,k) \cdot deltat/(ep0 \cdot ep(i,j,k))) \cdot E_{yx}(i,j,k,2) + ((1 - \exp(- zigmax(i,j,k) \cdot deltat)/(ep0 \cdot ep(i,j,k))))/(zigmax(i,j,k) \cdot delta) \cdot (Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3));
\]

end

if \( k = PD + 1 \)

E_{zx}(i,j,k,3) = E_{yx}(i,j,k,2) + ((deltat/(ep0 \cdot ep(i,j,k))) \cdot (Hzx(i,j,k,3) + Hzy(i,j,k,3) - Hzx(i,j-1,k,3) - Hzy(i,j-1,k,3));
\]

end

end

if \( i = PD + 1 \) \&\& \( j = PD + 1 \) \&\& \( k = PD + 1 \)

E_{zx}(i,j,k,3) = \exp(- zigmax(i,j,k) \cdot deltat/(ep0 \cdot ep(i,j,k))) \cdot E_{zx}(i,j,k,2) + ((1 - \exp(- zigmax(i,j,k) \cdot deltat)/(ep0 \cdot ep(i,j,k))))/(zigmax(i,j,k) \cdot delta) \cdot (Hyz(i,j,k,3) + Hyx(i,j,k,3) - Hyz(i,j-1,k,3) - Hyx(i,j-1,k,3));
\]

end

end

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Exz(i,j,k,3)=Exz(i,j,k,2)-((deltat/(ep0*ep(i,j,k)))/delta)*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end
if i~=1 & k~=PD+1
Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
Eyx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2)-((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
end
if i~=1 && (i<=PD+1 && j==N-PD)
Ezx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)-((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
Ezy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)-((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j-1,k,3)-Hxz(i,j-1,k,3));
end
if i~=1 && (i<PD+1 && j==N-PD)
Ezx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j-1,k,3)-Hyx(i,j-1,k,3));
Ezy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j-1,k,3)-Hxz(i,j-1,k,3));
end
if k>=P-PD && i<=PD+1 && j>=PD+1 && j<N-PD
Efx(i,j,k,3)=Exy(i,j,k,3)-((deltat/(ep0*ep(i,j,k)))/delta)*(Hxz(i,j,k,3)+Hzy(i,j,k,3)-Hxz(i,j-1,k,3)-Hzy(i,j-1,k,3));
Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
end
if i~=1 && (i<=PD+1 && k>P-PD)
Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
Eyx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2)-((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
end
if i~=1 && (i<PD+1 && k==P-PD)
Eyz(i,j,k,3)=Eyz(i,j,k,2)+((deltat/(ep0*ep(i,j,k)))/delta)*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
Ezx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)-((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*delta))*(Hyz(i,j,k,3)+Hx(i,j,k,3)-Hyz(i,j-1,k,3)-Hx(i,j-1,k,3));
end
if i~=1 && (i<PD+1 && k>P-PD)
\[ 
\text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k)) \cdot (\text{Hxy}(i,j,k) + \text{Hzx}(i,j,k) - \text{Hxy}(i,j,k-1) - \text{Hzx}(i,j,k-1));
\]

\[ 
\text{Eyx}(i,j,k) = \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Eyx}(i,j,k,2) - ((1 - \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k) \cdot \text{deltat}))) \cdot (\text{Hzx}(i,j,k) + \text{Hyx}(i,j,k) - \text{Hyz}(i,j,k) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3)) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3));
\]

\[ 
\text{Hxyz}(i,j,k,3) = \text{Hzx}(i,j,k,3) + \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3); 
\]

\[ 
\text{Eyx}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Eyx}(i,j,k,2) - ((1 - \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k) \cdot \text{deltat}))) \cdot (\text{Hzx}(i,j,k) + \text{Hyx}(i,j,k) - \text{Hyz}(i,j,k) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3); 
\]

\[ 
\text{Eyz}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Eyz}(i,j,k,2) - ((1 - \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k) \cdot \text{deltat}))) \cdot (\text{Hzx}(i,j,k) + \text{Hyx}(i,j,k) - \text{Hyz}(i,j,k) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3); 
\]

\[ 
\text{Ezx}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))) \cdot \text{Ezx}(i,j,k,2) - ((1 - \exp(- \text{zigmax}(i,j,k) \cdot \text{deltat}/(\text{ep0} \cdot \text{ep}(i,j,k))))/(\text{zigmax}(i,j,k) \cdot \text{deltat}))) \cdot (\text{Hzx}(i,j,k) + \text{Hyx}(i,j,k) - \text{Hyz}(i,j,k) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3) - \text{Hzx}(i,j,k,3) - \text{Hyz}(i,j,k,3); 
\]
if \( k \geq P - PD \) && \( i \geq M - PD \) && \( j \geq PD + 1 \) && \( j < N - PD \)

if \( j = PD + 1 \)

\[ \text{Exy}(i,j,k,3) = \text{Exy}(i,j,k,2) + ((\text{deltat}/(ep0*ep(i,j,k))) / \delta) * (\text{Hzx}(i,j,k,3) + \text{Hzy}(i,j,k,3)) - \text{Hzx}(i-j-1,k,3) - \text{Hzy}(i-j-1,k,3); \]

\[ \text{Exz}(i,j,k,3) = \exp(-zigmax(i,j,k)*\text{deltat}/(ep0*ep(i,j,k))) * \text{Exz}(i,j,k,2) - ((1-\exp(-zigmax(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))) / (zigmax(i,j,k)*\delta)) * (\text{Hyz}(i,j,k,3) + \text{Hyx}(i,j,k,3)) - \text{Hyz}(i-j-1,k,3) - \text{Hyx}(i-j-1,k,3)); \]

end

if \( (i = M - PD \) && \( k = P - PD \))

\[ \text{Eyz}(i,j,k,3) = \exp(-zigmaz(i,j,k)*\text{deltat}/(ep0*ep(i,j,k))) * \text{Eyz}(i,j,k,2) + ((1-\exp(-zigmaz(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))) / (zigmaz(i,j,k)*\delta)) * (\text{Hxy}(i,j,k,3) + \text{Hxz}(i,j,k,3)) - \text{Hxy}(i-j-1,k,3) - \text{Hxz}(i-j-1,k,3)); \]

end

if \( (j = N - PD \) && \( k = P - PD \))

\[ \text{Exy}(i,j,k,3) = \exp(-zigmay(i,j,k)*\text{deltat}/(ep0*ep(i,j,k))) * \text{Exy}(i,j,k,2) + ((1-\exp(-zigmay(i,j,k)*\text{deltat}/(ep0*ep(i,j,k)))) / (zigmay(i,j,k)*\delta)) * (\text{Hxz}(i,j,k,3) + \text{Hzy}(i,j,k,3)) - \text{Hxz}(i-j-1,k,3) - \text{Hzy}(i-j-1,k,3)); \]

end

end
Exz(i,j,k,3) = \exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-((1-\exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat))*(Hyz(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3))
end
if i~=PD+1
Eyz(i,j,k,3) = \exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2)+((1-\exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
end
if j~=1 && k~=1 && i~=PD+1
Exy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k-1,3)-Hzy(i,j,k-1,3));
end
if i~=1 && k~=1 && j~=PD+1
Ezx(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end

% Applying PML conditions to the 8 corners
% % 4 Bottom Corners %
if (k<=PD+1 && i<=PD+1 && j<=PD+1)
  if j-1 && k-1 && i-1
    Exy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k-1,3)-Hzy(i,j,k-1,3));
  end
  if i-1 && k-1 && j-1
    Ezy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k-1,3)-Hzy(i,j,k-1,3));
  end
end
if i-1 && k-1 && j-1
  Ezx(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end
if i-1 && j-1 && k-1
  Exz(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end

% % 4 Top Corners %
if (k>=PD+1 && i>=PD+1 && j>=PD+1)
  if j+1 && k+1 && i+1
    Exy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k+1,3)-Hzy(i,j,k+1,3));
  end
  if i+1 && k+1 && j+1
    Ezy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k+1,3)-Hzy(i,j,k+1,3));
  end
end
if i+1 && k+1 && j+1
  Ezx(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k+1,3)-Hyx(i,j,k+1,3));
end
if i+1 && j+1 && k+1
  Exz(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k+1,3)-Hyx(i,j,k+1,3));
end

% % 4 Right Corners %
if (k<=PD+1 && i>=PD+1 && j+1)
  if j+1 && k-1 && i-1
    Exy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k-1,3)-Hzy(i,j,k-1,3));
  end
  if i-1 && k-1 && j+1
    Ezy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k-1,3)-Hzy(i,j,k-1,3));
  end
end
if i-1 && k-1 && j+1
  Ezx(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end
if i-1 && j+1 && k-1
  Exz(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end

% % 4 Left Corners %
if (k>=PD+1 && i<=PD+1 && j-1)
  if j-1 && k+1 && i-1
    Exy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k-1,3)-Hzy(i,j,k-1,3));
  end
  if i-1 && k+1 && j-1
    Ezy(i,j,k,3) = \exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-\exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j,k-1,3)-Hzy(i,j,k-1,3));
  end
end
if i-1 && k+1 && j-1
  Ezx(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end
if i-1 && j-1 && k+1
  Exz(i,j,k,3) = \exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)+((1-\exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
end

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end
if (k<=PD+1 && i>=M-PD && j<=PD+1)
    if j=1 && k=1
        Exy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2) + ((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmay(i,j,k)*delta)))*(Hzx(i,j,k,3)+Hzx(i,j,k,3) - Hzx(i,j,k-1,3) - Hzy(i,j,k,3));
    Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2) + ((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta)))*(Hyz(i,j,k,3)+Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hzx(i,j,k,3));
end if i=1 && j=1 && k=1
    Eyz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2) + ((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta)))*(Hxy(i,j,k,3)+Hxz(i,j,k,3) - Hxy(i,j,k-1,3) - Hzx(i,j,k,3));
    Eyx(i,j,k,3)=exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2) + ((1-exp(-zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmax(i,j,k)*delta)))*(Hzx(i,j,k,3)+Hzy(i,j,k,3) - Hzx(i,j,k-1,3) - Hzy(i,j,k,3));
end if k=1
% ch j=1
    Exy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2) + ((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmay(i,j,k)*delta)))*(Hzx(i,j,k,3)+Hzx(i,j,k,3) - Hzx(i,j,k-1,3) - Hzy(i,j,k,3));
    Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2) + ((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta)))*(Hyz(i,j,k,3)+Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hzx(i,j,k,3));
end if i=1 && j=1 && k=1
end if (k<=PD+1 && i>=M-PD && j<=PD+1)
if j=1 && k=1
    Exy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2) + ((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmay(i,j,k)*delta)))*(Hzx(i,j,k,3)+Hzx(i,j,k,3) - Hzx(i,j,k-1,3) - Hzy(i,j,k,3));
    Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2) + ((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta)))*(Hyz(i,j,k,3)+Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hzx(i,j,k,3));
end if i=1 && k=1
% ch j=1
    Exy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2) + ((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmay(i,j,k)*delta)))*(Hzx(i,j,k,3)+Hzx(i,j,k,3) - Hzx(i,j,k-1,3) - Hzy(i,j,k,3));
    Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2) + ((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta)))*(Hyz(i,j,k,3)+Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hzx(i,j,k,3));
end if k=1
% ch i=1 && j=1
    Exy(i,j,k,3)=exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2) + ((1-exp(-zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmay(i,j,k)*delta)))*(Hzx(i,j,k,3)+Hzx(i,j,k,3) - Hzx(i,j,k-1,3) - Hzy(i,j,k,3));
    Exz(i,j,k,3)=exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2) + ((1-exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta)))*(Hyz(i,j,k,3)+Hyx(i,j,k,3) - Hyz(i,j,k-1,3) - Hzx(i,j,k,3));
end if (k<=PD+1 && i>=M-PD && j<=PD+1)
}
\[ \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hyx}(i,j,k,3) - \text{Hyx}(i-1,j,k,3) - \text{Hyx}(i-1,j,k,3)) ; \]

\[ \text{Ezy}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Ezy}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Ezx}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Ezx}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Eyz}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Eyz}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Eyx}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Eyx}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Ezx}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Ezx}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Eyz}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Eyz}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Eyx}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Eyx}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Ezx}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Ezx}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]

\[ \text{Eyz}(i,j,k,3) = \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times \text{Eyz}(i,j,k,2) - (1 - \exp(- \text{zigmax}(i,j,k) \times \frac{\text{deltat}}{(\varepsilon_0 \cdot \varepsilon(i,j,k)})) \times (\text{Hzx}(i,j,k,3) + \text{Hzx}(i,j,k,3) - \text{Hzx}(i-1,j,k,3) - \text{Hzx}(i-1,j,k,3)) ; \]
zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))/(zigmaz(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
  Exy(i,j,k,3)=exp(-
    zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2)-((1-exp(-
    zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i-1,j,k,3)-Hzy(i-1,j,k,3));
end
  if i-1 && j-1
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-
      exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
    Ezy(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-
      exp(-zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
  end
end
if (k>=P-PD && i>=M-PD && j<=PD+1)
  if j-1
    Exy(i,j,k,3)=exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2)+((1-exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hzx(i,j,k,3)+Hzy(i,j,k,3)-Hzx(i,j-1,k,3)-Hzy(i,j-1,k,3));
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
    Ezy(i,j,k,3)=exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
    Exy(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
  if j-1
    Exy(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
    Ezy(i,j,k,3)=exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
    Exy(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
end
if (k>=P-PD && i>=M-PD && j>=N-PD)
  if j-PD+1
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
    Ezy(i,j,k,3)=exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
    Exy(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
  if j-1
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
end
if (k>=P-PD && i>=M-PD && j>=N-PD)
  if j-PD+1
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
    Ezy(i,j,k,3)=exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
    Exy(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
  if j-1
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
end
if (k>=P-PD && i>=M-PD && j>=N-PD)
  if j-PD+1
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
    Ezy(i,j,k,3)=exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+((1-exp(-
      zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*delta))*(Hxy(i,j,k,3)+Hxz(i,j,k,3)-Hxy(i,j,k-1,3)-Hxz(i,j,k-1,3));
    Exy(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
  if j-1
    Ezx(i,j,k,3)=exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezx(i,j,k,2)+((1-exp(-
      zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*delta))*(Hyz(i,j,k,3)+Hyx(i,j,k,3)-Hyz(i,j,k-1,3)-Hyx(i,j,k-1,3));
  end
endif
zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))/((zigmay(i,j,k)*deltat)/(ep0*ep(i,j,k)))*(Hzx(i,j,k,3)-Hzx(i,j-k,3)-Hzy(i,j-k,3)-Hzy(i,j-1,k,3));

Exz(i,j,k,3)=exp(-
zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)-(1-exp(-
zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hyx(i,j-k,3)-Hzx(i,j-k,3)-Hyx(i,j-k,3));

Eyz(i,j,k,3)=exp(-
zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyz(i,j,k,2)-(1-exp(-
zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hyx(i,j-k,3)-Hzx(i,j-k,3)-Hyx(i,j-k,3));

Eyx(i,j,k,3)=exp(-
zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Eyx(i,j,k,2)+(1-exp(-
zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hyx(i,j-k,3)-Hzx(i,j-k,3)-Hyx(i,j-k,3));

end

if (k>=P-PD && i<=PD+1 && j>=N-PD)

if i=PD+1

Exy(i,j,k,3)=exp(-
zigmay(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exy(i,j,k,2)+(1-exp(-
zigmay(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmay(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hyx(i,j-k,3)-Hzx(i,j-k,3)-Hyx(i,j-k,3));

Exz(i,j,k,3)=exp(-
zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k)))*Exz(i,j,k,2)+(1-exp(-
zigmaz(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmaz(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hyx(i,j-k,3)-Hzx(i,j-k,3)-Hyx(i,j-k,3));

Ezy(i,j,k,3)=exp(-
zigmax(i,j,k)*deltat/(ep0*ep(i,j,k)))*Ezy(i,j,k,2)+(1-exp(-
zigmax(i,j,k)*deltat/(ep0*ep(i,j,k))))/(zigmax(i,j,k)*deltat))*(Hzx(i,j,k,3)+Hyx(i,j-k,3)-Hzx(i,j-k,3)-Hyx(i,j-k,3));

end

end

end
%%% Starting near-to-far field transformation %%%

for i=PD+2:1:M-PD-2
    for j=PD+2:1:N-PD-2
        for k=PD+2:1:P-PD-2
            Rp=[i j k]-cvector)*delta;
            if i==PD+2
                DiT=(sqrt(R*R')-Rp*RU')/(c*deltat);
                nn=fix(n+DiT);
                a=n+DiT-nn;
                if a>=0 & & a<=1
                    U1z(nn)=U1z(nn)+((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ey(i,j,k,2));
                    W1z(nn)=W1z(nn)-((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hy(i,j,k,3)-Hy(i,j,k,2));
                    U1y(nn)=U1y(nn)+((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ez(i,j,k,3)-Ez(i,j,k,2));
                    W1y(nn)=W1y(nn)-((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hz(i,j,k,3)-Hz(i,j,k,2));
                end
            end
            U1z(nn)=U1z(nn)+((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ey(i,j,k,2));
            W1z(nn)=W1z(nn)-((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hy(i,j,k,3)-Hy(i,j,k,2));
            U1y(nn)=U1y(nn)+((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ez(i,j,k,3)-Ez(i,j,k,2));
            W1y(nn)=W1y(nn)-((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hz(i,j,k,3)-Hz(i,j,k,2));
        end
    end
end

if i==M-PD-2
    DiT=(sqrt(R*R')-Rp*RU')/(c*deltat);
    nn=fix(n+DiT);
    a=n+DiT-nn;
    if a>=0 & & a<=1
        U2z(nn)=U2z(nn)-((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ey(i,j,k,2));
        W2z(nn)=W2z(nn)+((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hy(i,j,k,3)-Hy(i,j,k,2));
        U2y(nn)=U2y(nn)+((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ez(i,j,k,3)-Ez(i,j,k,2));
        W2y(nn)=W2y(nn)-((1-a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hz(i,j,k,3)-Hz(i,j,k,2));
    end
end

if j==PD+2
    DiT=(sqrt(R*R')-Rp*RU')/(c*deltat);
    nn=fix(n+DiT);
    a=n+DiT-nn;
    if a>=0 && a<=1
        U3z(nn)=U3z(nn) - ((1-
            a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ex(i,j,k,3)-Ex(i,j,k,2));
        U3z(nn+1)=U3z(nn+1) - ((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ex(i,j,k,2));
    end
    W3z(nn)=W3z(nn) + ((1-
        a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hz(i,j,k,3)-Hx(i,j,k,2));
    W3z(nn+1)=W3z(nn+1) + ((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hz(i,j,k,3)-Hx(i,j,k,2));
end

if k==PD+2
    DiT=(sqrt(R*R')-Rp*RU')/(c*deltat);
    nn=fix(n+DiT);
    a=n+DiT-nn;
    if a>=0 && a<=1
        U3x(nn)=U3x(nn) - ((1-
            a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ez(i,j,k,3)-Ez(i,j,k,2));
        U3x(nn+1)=U3x(nn+1) - ((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ez(i,j,k,3)-Ez(i,j,k,2));
    end
    W3x(nn)=W3x(nn) + ((1-
        a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hz(i,j,k,3)-Hz(i,j,k,2));
    W3x(nn+1)=W3x(nn+1) + ((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hz(i,j,k,3)-Hz(i,j,k,2));
end
if a>=0 && a<=1
U5z(nn)=U5z(nn)-((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ex(i,j,k,3)-Ex(i,j,k,2));
U5z(nn+1)=U5z(nn+1)-
((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ex(i,j,k,2));
W5z(nn)=W5z(nn)+((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hx(i,j,k,3)-Hx(i,j,k,2));
W5z(nn+1)=W5z(nn+1)+((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hx(i,j,k,3)-Hx(i,j,k,2));
U5x(nn)=U5x(nn)+((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ey(i,j,k,2));
U5x(nn+1)=U5x(nn+1)+((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ey(i,j,k,2));
W5x(nn)=W5x(nn)-((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hx(i,j,k,3)-Hx(i,j,k,2));
W5x(nn+1)=W5x(nn+1)-
((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hx(i,j,k,3)-Hx(i,j,k,2));
end
end
end
end

if k==P-PD-2
DiT=(sqrt(R*R')-Rp*RU')/(c*deltat);
nn=fix(n+DiT);
a=n+DiT-nn;
if a>=0 && a<=1
U6z(nn)=U6z(nn)+((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ex(i,j,k,3)-Ex(i,j,k,2));
U6z(nn+1)=U6z(nn+1)+((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ex(i,j,k,2));
W6z(nn)=W6z(nn)-((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hx(i,j,k,3)-Hx(i,j,k,2));
W6z(nn+1)=W6z(nn+1)-
((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hx(i,j,k,3)-Hx(i,j,k,2));
U6x(nn)=U6x(nn)-((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ey(i,j,k,2));
U6x(nn+1)=U6x(nn+1)-
((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Ey(i,j,k,3)-Ey(i,j,k,2));
W6x(nn)=W6x(nn)+((1-
a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hy(i,j,k,3)-Hy(i,j,k,2));
W6x(nn+1)=W6x(nn+1)+((a)*delta^2/(4*pi*sqrt(R*R')*c*deltat))*(Hy(i,j,k,3)-Hy(i,j,k,2));
end
end
end
end

Ux(nn)=U3x(nn)+U4x(nn)+U5x(nn)+U6x(nn);
Uy(nn)=U1y(nn)+U2y(nn)+U5y(nn)+U6y(nn);
Uz(nn)=U1z(nn)+U2z(nn)+U3z(nn)+U4z(nn);
Wx(nn)=W3x(nn)+W4x(nn)+W5x(nn)+W6x(nn);
Wy(nn)=W1y(nn)+W2y(nn)+W5y(nn)+W6y(nn);
Wz(nn)=W1z(nn)+W2z(nn)+W3z(nn)+W4z(nn);

Wtheta(nn)=Wx(nn)*cos(thetap)*cos(phi)+Wy(nn)*cos(thetap)*sin(phi)-
Wz(nn)*sin(thetap);
Wphi(nn)=Wx(nn)*sin(phi)+Wy(nn)*cos(phi);
Utheta(nn)=Ux(nn)*cos(thetap)*cos(phi)+Uy(nn)*cos(thetap)*sin(phi)-
Uz(nn)*sin(thetap);
Uphi(nn)=Ux(nn)*sin(phi)+Uy(nn)*cos(phi);

Etheta(nn)=-etha0*Wtheta(nn)-Uphi(nn);
Ephi(nn)=-etha0*Wphi(nn)+Utheta(nn);

%%%%%%%%%%%%%%%%

Hx(:,:,1)=Hx(:,:,2);
Hx(:,:,2)=Hx(:,:,3);
Hy(:,:,1)=Hy(:,:,2);
Hy(:,:,2)=Hy(:,:,3);
Hz(:,:,1)=Hz(:,:,2);
Hz(:,:,2)=Hz(:,:,3);
Ex(:,:,1)=Ex(:,:,2);
Ex(:,:,2)=Ex(:,:,3);
Ey(:,:,1)=Ey(:,:,2);
Ey(:,:,2)=Ey(:,:,3);
Ez(:,:,1)=Ez(:,:,2);
Ez(:,:,2)=Ez(:,:,3);

Ex(25,25,25,3);
Ey(25,:,25,3);
Ez(25,:,25,3)

Hxy(:,:,1)=Hxy(:,:,2);
Hxy(:,:,2)=Hxy(:,:,3);
Hyx(:,:,1)=Hyx(:,:,2);
Hyx(:,:,2)=Hyx(:,:,3);
Hzx(:,:,1)=Hzx(:,:,2);
Hzx(:,:,2)=Hzx(:,:,3);
Exy(:,:,1)=Exy(:,:,2);
Exy(:,:,2)=Exy(:,:,3);
Eyx(:,:,1)=Eyx(:,:,2);
Eyx(:,:,2)=Eyx(:,:,3);
Ezx(:,:,1)=Ezx(:,:,2);
Ezx(:,:,2)=Ezx(:,:,3);

Hxy(25,:,25,3);

Hxz(:,:,1)=Hxz(:,:,2);
Hxz(:,:,2)=Hxz(:,:,3);
Hyz(:,:,1)=Hyz(:,:,2);
Hyz(:,:,2)=Hyz(:,:,3);
Hzy(:,:,1)=Hzy(:,:,2);
Hzy(:,:,2)=Hzy(:,:,3);
Exz(:,:,1)=Exz(:,:,2);
Exz(:,:,2)=Exz(:,:,3);
Eyz(:,:,1)=Eyz(:,:,2);
Eyz(:,:,2)=Eyz(:,:,3);
Ezy(:,:,1)=Ezy(:,:,2);
Ezy(:,:,2)=Ezy(:,:,3);
Erx(:,1)=Erx(:,2); Erx(:,2)=Erx(:,3);
Ery(:,1)=Ery(:,2); Ery(:,2)=Ery(:,3);
Erz(:,1)=Erz(:,2); Erz(:,2)=Erz(:,3);
Hrx(:,1)=Hrx(:,2); Hrx(:,2)=Hrx(:,3);
Hry(:,1)=Hry(:,2); Hry(:,2)=Hry(:,3);
Hrz(:,1)=Hrz(:,2); Hrz(:,2)=Hrz(:,3);
Dx(:,:,1)=Dx(:,:,2);
Dx(:,:,2)=Dx(:,:,3);
Dy(:,:,1)=Dy(:,:,2);
Dy(:,:,2)=Dy(:,:,3);
Dz(:,:,1)=Dz(:,:,2);
Dz(:,:,2)=Dz(:,:,3);
Ezy(25,:,:,:,3);
Ezy(25,:,:,:,3)+Ezx(25,:,:,:,3)
Ey(25,:,:,:,3)
Ex(25,:,:,:,3)
Einci(n)=Erz(sref+irr,3);
EEpe(n)=pe;
EE(n)=Ez(25,31,25,3)
EE2(n)=Ez(25,34,25,3)
EE4(n)=Ez(25,36,25,3)
EE6(n)=Ez(25,14,25,3)
if mod(n,2)==0
    nsnap=n/2;
    Esnap(:,:,nsnap)=Ez(:,:,3);
end
for j=1:1:N-1
    for k=1:1:P-1
        EE5(k,j,nsnap)=Esnap(25,j,k,nsnap);
    end
end
j=1:1:N-1;
k=1:1:P-1;
pause on
hold on
figure(3),surface(j,k,EE5(j,k,nsnap));
drawnow, pause (0.05)
pause off
end
% end of time loop
Ezy(25,13,25,3)=0;
Ezx(25,13,25,3)=0;
Ezy(25,37,25,3)=0;
Ezx(25,37,25,3)=0;
i=1:1:N;
figure(1), plot(i, 10*log10(abs(Ez(25,:,25,3))));
hold on
figure(1), plot(i, 10*log10(abs(Ezy(25,:,25,3)+Ezx(25,:,25,3)+Ez(25,:,25,3))),'b');
hold on
figure(1), plot(i, 10*log10(abs(Ex(25,:,25,3))),'r');
hold on
figure(1), plot(i, 10*log10(abs(Exy(25,:,25,3)+Exy(25,:,25,3)+Ex(25,:,25,3))),'r');

j=1:1:nmax;
figure(5), plot(j, EEpe(j), 'r');
figure(2), subplot(211), plot(j*deltat, (EE(j)),'g');
Ezfft=fft(EE);
hold on
grid on
figure(2), subplot(212), semilogx((j-1)/(nmax*deltat),10*log10(deltat*abs(Ezfft(j))));
EE1=EE';
save('EE.txt', 'EE1', '-ascii')
xlswrite('EE.xls', EE', 'A1:A1500')
hold on
figure(2), subplot(211), plot(j*deltat, (EE2(j)),'r');
Ezfft=fft(EE2);
figure(2), subplot(212), semilogx((j-1)/(nmax*deltat),10*log10(deltat*abs(Ezfft(j))),'r');
EE3=EE2';
legend('EE', 'EE2')
Eincifft=fft(Einci);
figure(2), subplot(212), semilogx((j-1)/(nmax*deltat),10*log10(deltat*abs(Eincifft(j))),'b');
c2=fix(DiT)+1;
mm=nn-c2
EEtheta=Etheta(c2:nn)
Ethetafft=fft(EEtheta);
figure(2), subplot(212), semilogx((j-1)/(nmax*deltat),10*log10(deltat*abs(Ethetafft(j))),'c');
EEphi=Ephi(c2:nn)
Ephifft=fft(EEphi);
figure(2), subplot(212), semilogx((j-1)/(nmax*deltat),10*log10(deltat*abs(Ephifft(j))),'k');
for i1=1:1:nmax
RCS(i1)=(4*pi*R1*R1')*(Ethetafft(i1)*conj(Ethetafft(i1))+Ephifft(i1)*conj(Ephifft(i1)))/(2*etha0*Eincifft(i1)*conj(Eincifft(i1)));
end
figure(6), semilogx((j-1)/(nmax*deltat),10*log10(abs(RCS(j))),'r')
figure(7), plot((j-1)/(nmax*deltat),10*log10(abs(RCS(j))),'r')
hold on
save('EE2.txt', 'EE3', '-ascii')
xlswrite('EE2.xls', EE2', 'A1:A1500')
save('EE4.txt', 'EE4', '-ascii')
xlswrite('EE4.xls',EE4,'A1:A1500')
save('EE6.txt','EE6','-ascii')
xlswrite('EE6.xls',EE6,'A1:A1500')

grid on
hold on
figure(2), subplot(211), plot(j*deltat,(EE4(j)),'b');
hold on
figure(2), subplot(211), plot(j*deltat,(EE6(j)),'k');

for n=1:1:nsnap
    for j=1:1:N-1
        for k=1:1:P-1
            EE5(k,j,n)=Esnap(25,j,k,n);
        end
    end
for n=1:1:nsnap
    j=1:1:N-1;
    k=1:1:P-1;
    pause on
    hold on
    figure(4), surface(j,k,abs(EE5(j,k,n)));
    hold off
    drawnow, pause (0.1)
    pause off
end
Vitae

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