The basic theory of varieties in algebraic geometry

by

Rasha Almohammadi

B.Sc., Taibah University, 2005

A REPORT SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

In the Graduate Academic Unit of Math and Stats

Supervisor(s):	Colin Ingalls,PhD/Math and Stats
Examining Board:	Alyssa Sankey, PhD, /Math and Stats
	Bahram Rangipour, PhD/Math and Stats

This report is accepted

Dean of Graduate Studies

THE UNIVERSITY OF NEW BRUNSWICK

September, 2016

Abstract

In algebraic geometry, a variety is a set of zeroes of a set of polynomial equations in an arbitrary finite number of variables. The order reversing correspondence between varieties and ideals establishes a bridge between the algebraic nature of polynomial rings and the geometry of affine varieties. For algebraically closed fields, Hilbert's Nullstellensatz states that this order reversing correspondence restricts to a one-to-one correspondence between varieties and radical ideals, between irreducible varieties and prime ideals, and between points and maximal ideals.

In this work, we shall discuss affine varieties, the Zariski topology (the topology where the closed sets are the affine varieties), coordinate rings, and morphisms between varieties.

Contents

Chapter 1

Rings and Ideals

We start with reviewing the definition and elementary properties of rings. After this we will be examining ideals. Ideals are algebraic objects which were first defined by Richard Dedekind in 1876. Later, the concept was expanded by David Hilbert. After Hilbert, Emmy Noether pushed the limit and expanded our understanding of ideals even further in her paper [3].

1.1 Rings

A commutative ring is an algebraic structure that satisfies the most commonly used properties of the integers or the polynomials considered with the operations of addition and multiplication. It is remarkable how much can be understood without the existence of multiplicative inverses.

Definition 1.1.1. *A ring is a triplet* $(R, +, \cdot)$ *with properties*

1. $(R, +)$ *is an abelian group (that is,* $(R, +)$ *is closed, associative, it has a zero* *element denoted by* 0*, every element* x *of* R *has an (additive) inverse which is denoted by* $-x$*, for all* $x \in R$*)*

2. $(R, \cdot)$ *is associative, that is*

   $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for any* a*,* b*,* c *in* R*.*

   Note: *From now on "$\cdot$" will be dropped when multiplying.*

3. *Multiplication is distributive over addition:*

   $a(b + c) = ab + ac$ *for any* a*,* b*,* c *in* R

   $(a + b)c = ac + bc$ *for any* a*,* b*,* c *in* R*.*

If multiplication is commutative: $ab = ba$ for any a,b in R, we say that R is a **commutative** ring.

If there exists an element $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$ for any a in R. We say that R is a **ring with identity**, It is not difficult to see that if R is a ring with identity, then the identity is unique.

A **Field** is a commutative ring with identity, in which all non-zero elements have multiplicative inverses.

We will consider only commutative rings.

Example 1.1.1. *The set of all integers* $\mathbb{Z}$*, under the usual addition and multiplication operations forms a commutative ring with identity.*

Example 1.1.2. $(\mathbb{R}[X], +, \cdot)$ *the set of all polynomials with coefficients in* $\mathbb{R}$ *under the usual addition and multiplication operations forms a commutative ring with identity.*

Example 1.1.3. $(\mathbb{R}_{n,n}, +, \cdot)$ *the set of all* $n \times n$ *amtrices with coefficients in* $\mathbb{R}$ *under the usual addition and multiplication operations forms a non commutative ring with identity.*

1.2 Ideals

Definition 1.2.1. *Let* R *be a commutative ring. A subset* $I \subseteq R$ *is said to be an* ***ideal*** *of* R *if* I *satisfies*

1. *If* $f, g \in I$*, then* $f + g \in I$.
2. *If* $f \in I$ *and* $h \in R$*, then* $hf \in I$.

Note 1.2.1. *In a ring* R*, the set* R *itself forms an ideal of* R*. Also, the subset containing only the additive identity* 0_R *forms an ideal. These two ideals are usually referred to as the trivial ideals of* R*.*

Example 1.2.2. *The set*

$$2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

is an ideal of $\mathbb{Z}$*, the ring of integers.*

In fact ideals of $\mathbb{Z}$ are exactly of the form $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$, where $n \in \mathbb{Z}$. One can consider ideals as a generalization of the notion of subgroups. Those who are familiar with group theory may notice that in many ways ideals are analogous to normal subgroups; for instance, both can be utilized to form quotients.

Let R be a ring, I an ideal of R. Define the equivalence relation $\sim$ on R as $a \sim b \Leftrightarrow a - b \in I$. One can check this is indeed an equivalence relation and its equivalence classes of $(R, \sim)$ would be of the form $r + I = \{r + a \mid a \in I\}$. This leads us to following definition. We will denote by $R/I := (R, \sim)$.

In R/I we define:

1. $(r + I) + (r' + I) = (r + r') + I$
2. $(r + I) \cdot (r' + I) = (rr') + I$

Note that since the operations in the preceding definition are defined on classes, we should make sure that the definition of operations is independent of choice of the class representative which means the above introduced operations are well-defined. More precisely we should show that if $r_1 + I = s_1 + I$ and $r_2 + I = s_2 + I$ then

1. $(r_1 + I) + (r_2 + I) = (s_1 + I) + (s_2 + I)$
2. $(r_1 + I) \cdot (r_2 + I) = (s_1 + I) \cdot (s_2 + I)$

or

1. $(r_1 + r_2) + I = (s_1 + s_2) + I$
2. $(r_1 \cdot r_2) + I = (s_1 \cdot s_2) + I$

which is easily achievable if we use the definition of the relation $\sim$ given above and the fact that I is an ideal of R.

The above tells us that R/I is a ring, called the **quotient ring** of R modulo I.

Example 1.2.3. *As a prototypical example of the integers, consider the quotient ring* $\frac{\mathbb{Z}}{2\mathbb{Z}}$. *Then,* $\frac{\mathbb{Z}}{2\mathbb{Z}} = \{[0], [1]\}$, *and with the quotient ring operations,* $\frac{\mathbb{Z}}{2\mathbb{Z}}$ *forms a ring. In fact* $\frac{\mathbb{Z}}{2\mathbb{Z}}$ *is a finite field of order* 2.

Definition 1.2.2. *A domain is a ring (assumed nonzero) in which* $ab = 0 \implies a = 0 \,\text{or}\, b = 0$. *Commutative domains are called* ***integral domains***.

Example 1.2.4. *All fields are integral domains.*

Definition 1.2.3. *Let* R *be a ring. An ideal* $I \subseteq R$ *is prime if* $xy \in I \Rightarrow x \in I$ *or* $y \in I$.

Proposition 1.2.5. *Let* R *be a ring and* $I \subseteq R$ *be an ideal. Then,*

I *is prime* $\iff \frac{R}{I}$ *is a domain.*

Definition 1.2.4. *Let* R *be a ring. An ideal generated by* $x_1, \ldots, x_n$ *is defined as*

$$\langle x_1, \ldots, x_n \rangle := \left\{ \sum_{i=1}^{n} r_i x_i s_i \mid r_i, s_i \in R \right\}.$$

If R is commutative, then

$$\langle x_1, \ldots, x_n \rangle = \left\{ \sum_{i=1}^{n} r_i x_i \mid r_i \in R \right\}.$$

Example 1.2.6. *Any ideal of $\mathbb{Z}$ generated by a prime number $p \in \mathbb{Z}$, is a prime ideal. Indeed*

If $ab \in \langle p \rangle$ then $\exists k \in \mathbb{Z}$ such that $ab = pk$.

Since $p \mid ab$ and p is prime, therefore $p \mid a$ or $p \mid b$. Thus $a = pk'$ or $b = pk''$ respectively for some $k', k'' \in \mathbb{Z}$.

Thus, $a \in \langle p \rangle$ or $b \in \langle p \rangle$. It means $I = \langle p \rangle$ is a prime ideal.

The above example tells us that every ideal of $\mathbb{Z}$ generated by a prime number is a prime ideal. In fact the notion of prime ideal is a natural generalization of prime numbers.

Definition 1.2.5. *A monomial in $x_1, \cdots, x_n$ is a product of the form*

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} \cdot x_n^{\alpha_n}$$

*where all exponents $\alpha_1, \cdots, \alpha_n$ are non-negative integers. The **total degree** of this* *monomial is the sum $\alpha_1 + \cdots + \alpha_n$.*

Definition 1.2.6. *A polynomial f in $x_1, \cdots, x_n$ with coefficients in R is a finite linear combination of monomials. We can write a polynomial f in the form*

$$f = \sum_{\alpha} a_\alpha X^\alpha \quad a_\alpha \in R$$

where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and by X^α we mean $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. The set of all possible polynomials in $x_1, \cdots, x_n$ with coefficients in R form a commutative ring denoted $R[x_1, \cdots, x_n]$.

Definition 1.2.7. ***A principal ideal*** *is an ideal I in a ring R that is generated by a single element a of R through multiplication by every element of R.*

Example 1.2.7. *Let $R = \mathbb{C}[x]$, the single variable complex polynomial ring. Then $\langle x - \lambda \rangle$ is a prime ideal for all $\lambda \in \mathbb{C}$.*

Proof. Consider the polynomial $R = \mathbb{C}[x]$. To prove that $I = \langle x - \lambda \rangle$ is a prime ideal of R, we prove that if $f(x)g(x) \in I$, then either $f(x) \in I$ or $g(x) \in I$.

Now we have $I = \langle x - \lambda \rangle = \{(x - \lambda)h(x) : h(x) \in R\}$. Let $f(x)g(x) \in I$, then by definition

$f(x)g(x) = (x - \lambda)h(x)$ for some $h(x) \in R$.

For $x = \lambda$, $f(\lambda) \cdot g(\lambda) = (\lambda - \lambda)h(\lambda) = 0$.

$\Rightarrow f(\lambda) = 0$ or $g(\lambda) = 0$. If $f(\lambda) = 0$,

then $f(x) = (x - \lambda)h_1(x)$ for some $h_1(x) \in R \Rightarrow f(x) \in I$.

If $g(\lambda) = 0$, then $g(x) = (x - \lambda)h_2(x)$ for some $h_2(x) \in R \Rightarrow g(x) \in I$.

Therefore, $f(x)g(x) \in I$ implies $f(x) \in I$ or $g(x) \in I$,

so by the definition of prime ideal, $I = \langle x - \lambda \rangle$ is a prime ideal of $R = \mathbb{C}[x]$.

$\square$

Definition 1.2.8. *An ideal $I \subsetneq R$ in a ring R is maximal if it is maximal among the proper ideals of R with respect to inclusion, i.e., For any ideal J with $I \subseteq J$, either $J = I$ or $J = R$.*

Thus I is maximal if and only if R/I is nonzero and has no proper nonzero ideals. We can observe that I is maximal if and only if R/I is a field.

Example 1.2.8. *Let F be a field. Then the only maximal ideal of F is* 0.

Proposition 1.2.9. *Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Then every prime ideal in A is maximal.*

Proof. Let P be a prime ideal of A. Therefore A/P is an integral domain. Given any non-zero $x + P \in A/P$, there will be a suitable $n \in \mathbb{N} - \{1\}$, such that $x^n = x$ or equivalently $x(x^{n-1} - 1) = 0 \in P$. Since P is prime and $x + P$ is non-zero therefore

$x^{n-1} - 1 \in P$. This implies that $x^{n-1} + P = 1 + P$. Since $n > 1$, $x^{n-1} + P = 1 + P$ we have $(x+P)(x^{n-2}+P) = 1+P$. Hence that $x+P$ is invertible. Therefore, A/P is a field and P is a maximal ideal.

$\square$

Let R be a commutative ring with identity, and $S \subset R$ a subset. We say that S is multiplicative if

1. $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$.
2. $0 \notin S$, $1 \in S$

We define the **localization of** R **with respect to** S and denoted R_S, as follows. Its elements are formal expressions of the form $\frac{a}{s}$, $a \in R$, $s \in S$, where $\frac{a_1}{s_1} = \frac{a_2}{s_2}$, if and only if, there exist $s \in S$, such that $s(a_1 s_2 - a_2 s_1) = 0$.

If we endow R_S with the usual operations of addition and multiplication of fractions, then R_S is a ring. The unit in R_S is $\frac{1}{1}$, and the zero is $\frac{0}{1}$.

Example 1.2.10. *If $a \in R$, let $S = \{a^n : n \in \mathbb{N}\}$. Then S is multiplicatively closed. In this case localization of R with respect to S is denoted by R_a, and is called the* ***localization of*** R ***at*** a***.***

Example 1.2.11. *If $P \subset R$ is a prime ideal, let $S = R - P$. Then S is multiplicatively closed. In this case localization of R with respect to S is denoted by R_P, and is called the* ***localization of*** R ***at*** P***.***

Definition 1.2.9. *The spectrum of a commutative ring R, denoted by Spec(R), is the set of all prime ideals of R. We also define maxSpec(A) to be the set of all maximal ideals of A.[4, page 84].*

Definition 1.2.10. *Given some algebraically closed field $\mathbb{K}$, algebraic sets are the subsets of $\mathbb{K}^n$ defined as common zeroes of $S \subset \mathbb{K}[x_1, ..., x_n]$.*

If $\mathcal{A}$ is such an algebraic set, then we consider the commutative ring R consisting of those polynomials of which $\mathcal{A}$ is the algebraic set.

The maximal ideals of R correspond to the points of $\mathcal{A}$, while the prime ideals of R correspond to the varieties in $\mathcal{A}$ (varieties are irreducible algebraic sets). Therefore, Spec(R) consists of the points and varieties in $\mathcal{A}$.

1.3 Noetherian Rings

Definition 1.3.1. *A ring R is Noetherian (or it is called a Noetherian ring) if for all ascending chains of proper ideals-embedding, say:*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

(For all j in the natural numbers, I_j are all ideals of R.) there is a positive integer n such that $I_n = I_{(n+1)}$; in other words, every ascending chain has a maximal element. This condition is sometimes called the A.C.C. (Ascending Chain Condition).

Proposition 1.3.1. *Suppose we have a ring R. Then, the following statements about R are equivalent:*

1. *R is Noetherian.*

2. *Any nonempty set of ideals in R contains a maximal element.*

Proof. $1 \Rightarrow 2$) Suppose that we have a nonempty set of ideals M without a maximal element. For each ideal $I_n \in M$ there exists an $I_{n+1} \in M$ such that $I_n \subsetneq I_{n+1}$. Therefore, one can construct a chain of ideals that is not stationary. But this contradicts our assumption.

$2 \Rightarrow 1$) Applying the maximal condition to the set of ideals in our chain of ideals, 1 follows.

□

In a Noetherian ring all ideals are finitely generated. Indeed, suppose there is an ideal I of R that is not finitely generated. For $r_i \in I$, $i = 1, \ldots, m$ by assumption $\langle r_1, r_2, \cdots, r_m \rangle \neq I$ for every $m \in \mathbb{N}$. Hence, there is $r_{m+1} \in I$ such that $r_{m+1} \notin \langle r_1, r_2, \cdots, r_m \rangle$. This would let us construct a chain of ideals $\langle r_1 \rangle \subset \langle r_1, r_2 \rangle \subset \cdots$ which never becomes stationary.

Note 1.3.2. *Every ring R with unity element 1_R is finitely generated as an ideal over itself. So every ring is finitely generated as an ideal, while it could have an ideal which is not finitely generated.*

1.4 Hilbert's Basis Theorem

Theorem 1.4.1. *Hilbert's Basis Theorem*

If R is a Noetherian ring, then $R[X]$ is a Noetherian ring.

Proof. This is a proof due to Heidrun Sarges [11], who proves that if $R[X]$ is not Noetherian, then R is not Noetherian. Let I be an ideal of $R[X]$ that is not finitely generated (of course I will be a non-zero ideal).

Using the ideal I, we will construct a sequence of ideals in R inductively, which can not become stationary. This contradicts the assumption of R being Noetherian.

Let $f_1 \in I$ be a polynomial of the least degree (we note that such f_1 exists since the set of degrees of polynomials in I is a subset of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and so it's well-ordered therefore it has a least element.)

Let $k \geq 1$ and assume f_k is chosen. Then let f_{k+1} be a polynomial of least degree in $I \setminus \langle f_1, \cdots, f_k \rangle$.

Let $a_k \in R$ be the leading coefficient of f_k and the non-negative integer d_k be the degree of f_k, for $k \geq 1$. Consider the ideal $J \subseteq R$ defined by $J = \langle a_k \mid k \in \mathbb{N} \rangle$. By the choice of f_k we have $d_1 \leq d_2 \leq \cdots$ and

$$\langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \langle a_1, a_2, a_3 \rangle \subseteq .... \tag{1.1}$$

Since R is Noetherian, the chain equation 1.1 will come to be stationary. Therefore, there is $n \in \mathbb{N}$ such that

$$\langle a_1, ..., a_n \rangle = \langle a_1, ..., a_{n+i} \rangle, \quad \text{for } i \geq 1. \tag{1.2}$$

Therefore $J = \langle a_i \mid i \in \mathbb{N} \rangle = \bigcup_{i \geq 1} \langle a_1, a_2, \dots, a_i \rangle = \langle a_1, a_2, \dots, a_n \rangle$.

Now let $I_0 = \langle f_1, f_2, \dots, f_n \rangle$. Then, by the choice of f_{n+1}, we have $f_{n+1} \notin I_0$. Since $J = \langle a_i \mid i \in \mathbb{N} \rangle = \langle a_1, a_2, \dots, a_n \rangle$, so $a_{n+1} \in \langle a_1, a_2, \dots, a_n \rangle$.

Therefore, there are $b_i \in R,\ i = 1, 2, \dots, n$ such that:

$$a_{n+1} = \sum_{i=1}^{n} b_i a_i \tag{1.3}$$

Now let $g = f_{n+1} - X^{d_{n+1}-d_1} b_1 f_1 \dots - X^{d_{n+1}-d_n} b_n f_n$. Then using the fact that I is an ideal and $f_1, \dots, f_{n+1} \in I$ we have $g \in I$.

By the way we defined g and using equation 1.3, we have $deg(g) < deg(f_{n+1})$.

Since $f_{n+1} \notin I_0$ and $f_i \in I_i$ for $i = 1, \dots n$, we have $g \notin I_0$ which contradicts the minimality of $deg(f_{n+1})$! (because $f_{n+1} \in J \backslash I_0$ is chosen in such a way that it has the lowest degree but $g \in J \backslash I_0$ and $deg(g) < deg(f_{n+1})$ which is contradiction).
so the assumption of $R\,[X]$ not being Noetherian is false.

□

An important geometric consequence of the Hilbert Basis Theorem is that every algebraic set is the zero set of a finite set of polynomials.

Corollary 1.4.2. *If R is Noetherian, then so is the polynomial ring $R[x_1, \dots, x_n]$.*

Proof. Complete the proof by induction on n using Hilbert's Basis Theorem.

□

Definition 1.4.1. *An element* $u \in R$ *for which there exists* $v \in R$ *such that* $uv = vu = 1$ *is called a unit.*

The set $R^* := \{\text{units of } R\}$ forms a group under multiplication.

We note that using Zorn's lemma, on can show every non-unit element of a ring is contained in a maximal ideal.

Definition 1.4.2. *A ring* R *is said to be local if it has exactly one maximal ideal* M.

Note 1.4.3. *For a local ring the set of unit elements of* R *is equal to* $R^* = R \backslash M$.

A ring homomorphism is a generalization of group homomorphism, which is a function between two rings that respects the algebraic structure.

Definition 1.4.3. *Let* R *and* S *be rings. A function* $f : R \to S$ *is a* ***ring homomorphism*** *if for all* $x, y \in R$

1. $f(x + y) = f(x) + f(y)$
2. $f(xy) = f(x)f(y)$
3. $f(1_R) = 1_S$

see [6,page 49].

Note 1.4.4. *Using $f(x+y) = f(x)+f(y)$ for $x = y = 0_R$ gives $f(0_R) = 0_S$. Therefore, $f(0_R) = 0_S$ automatically holds in all ring homomorphisms.*

Let R be a ring. $n \in \mathbb{N}$ is called the **characteristic** of R and it is denoted by $char(R)$, if n is the smallest positive integer such that $na = \underbrace{a + \cdots + a}_{n-\text{many}} = 0$ for all $a \in R$. If such $n \in \mathbb{N}$ does not exist we say that the ring R has characteristic 0.

Example 1.4.5. *Let R be a commutative ring with $char(R) = 2$. Define $f : R \longrightarrow R$ by $f(x) = x^2$, for all $x \in R$. Then f is a ring homomorphism.*

Proof. Let $a, b \in R$. Remember that $2x = 0$ for all $x \in R$, since $char(R) = 2$. Then $f(a + b) = (a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 = f(a) + f(b)$, and $f(ab) = (ab)^2 = a^2b^2 = f(a)f(b)$, so f respects addition and multiplication. Finally, $f(1) = 1^2 = 1$ $\square$

Proposition 1.4.6. *There is no ring homomorphism $\mathbb{C} \longrightarrow \mathbb{R}$.*

Proof. Suppose that there exists a ring homomorphism $f : \mathbb{C} \longrightarrow \mathbb{R}$.
Recall that, by definition of a ring homomorphism, we have $f(1_R) = 1_S$. Hence $f(-1_\mathbb{C}) = -1_\mathbb{R}$. Since $f : \mathbb{C} \longrightarrow \mathbb{R}$ is a also group homomorphism between the additive abelian groups of $\mathbb{C}$ and $\mathbb{R}$. Let $r = f(i) \in \mathbb{R}$. Since $i^2 = -1$ in $\mathbb{C}$ and f is a ring homomorphism, we have

$$-1_\mathbb{R} = f(-1_\mathbb{C}) = f(i^2) = f(i)^2 = r^2.$$

Thus $r \in \mathbb{R}$ is a real number such that $r^2 = -1$. This is a contradiction since for every $r \in \mathbb{R}$ we have $r^2 \geqslant 0$. Hence there is no ring homomorphism $f : \mathbb{C} \longrightarrow \mathbb{R}$.

$\square$

Chapter 2

Algebraic Varieties

The goal of algebraic geometry is to study solutions of polynomial equations in several variables over a fixed ground field. In this chapter we will exploring algebraic varieties, their definitions and basic properties.

Varieties are the fundamental objects used in the study of algebraic geometry. A variety is the set of common solutions of a set of polynomials. While a simple notion, it is also very powerful.

Algebraic varieties are at the very center of algebraic geometry. Algebraic varieties are the sets of solutions to a system of polynomial equations over the real or complex numbers. For those familiar with the idea of an analytic manifold, the idea of an algebraic variety is very similar, with the big difference being in that a variety can have singular points, while a manifold cannot.

2.1 Affine Spaces

Definition 2.1.1. *Let $\mathbb{K}$ be a field. An affine space of dimension n, $\mathbb{K}^n$ is the set of all n-tuples of elements in the field $\mathbb{K}$. In particular, $\mathbb{K}^1$ is called the affine line, and $\mathbb{K}^2$ is called the affine plane. The elements of $\mathbb{K}^n$ are called points which* ***we denoted by*** *capital leters.*

Example 2.1.1. *Consider the case $\mathbb{K} = \mathbb{R}$. Here we get the familiar space $\mathbb{R}^n$ from calculus and linear algebra.*

Let $f \in \mathbb{K}[x_1, x_2, ..., x_n]$ where $\mathbb{K}$ is a field. If $P = (a_1, a_2, ..., a_n) \in \mathbb{K}^n$, then, we define $f(P) = f(a_1, a_2, ..., a_n)$.

Definition 2.1.2. *Let there be a subset $V \subseteq \mathbb{K}^n$. V is called an* ***affine algebraic set*** *if there is a set $S \subseteq \mathbb{K}[x_1, x_2, ...x_n]$ such that*

$$V = Z(S) = \{P \in \mathbb{K}^n : f(P) = 0, \forall f \in S\}.$$

In words, one could say an algebraic set is the zero locus of a set of polynomials. Understanding it this way can be very helpful when speaking about algebraic sets and their uses. This definition is very similar to the definition of algebraic varieties, with good reason. An algebraic set is a "reducible" algebraic variety. The notion of reducibility of varieties is intertwined with the idea of the "Zariski" topology, which lets us say a variety is "irreducible" when it is not the union of two smaller closed subsets in the topology.

Example 2.1.2. *1. For any $a, b \in \mathbb{K}, \{(a,b)\}$ is an algebraic set in $\mathbb{K}^2$ since*

$$\{(a,b)\} = Z(x-a, y-b).$$

2. Any single point in $\mathbb{K}^n$ is an algebraic set:

$$Z(a_1, ..., a_n) = Z(x_1 - a_1, ..., x_n - a_n).$$

Proposition 2.1.3. *1. The union of two algebraic sets is an algebraic set.*

2. The intersection of any family of algebraic sets is an algebraic set.

3. The empty set and the whole space are algebraic sets.

Proof. 1. Let $V_1 = Z(S_1)$ and $V_2 = Z(S_2)$. If $P \in V_1 \cup V_2$, then either $P \in V_1$ or $P \in V_2$.

Thus, P is a zero of every polynomial in $S1 \cdot S2 = \{fg | f \in S_1 \text{ and } g \in S_2\}$ and $V_1 \cup V_2 \subseteq Z(S_1 \cdot S_2)$.

Conversely, if $P \in Z(S_1 \cdot S_2)$, and $P \notin V_1$, then there exist an $f \in S_1$ such that $f(P) \neq 0$. Since for any $g \in S_2$, $(f \cdot g)(P) = 0$ we have that $g(P) = 0$, so that $P \in V_2$.

2. If $V_i = Z(S_i)$ is any family of algebraic sets, then $\bigcap V_i = Z(\cup S_i)$

Thus, $\bigcap V_i$ is also an algebraic set.

3. Finally, the empty set $\emptyset = Z(1)$, and the whole space $\mathbb{K}^n = Z(0)$.

$\square$

The above proposition tells us that algebraic sets meet the conditions of the closed sets of a topology in $\mathbb{K}^n$. This topology is called the **Zariski topology** on $\mathbb{K}^n$.

Definition 2.1.3. *We say that a field* $\mathbb{K}$ *is* ***algebraically closed*** *if every non-constant polynomial* $f \in \mathbb{K}[X]$ *has a root in* $\mathbb{K}$. *i.e., there exists* $a \in \mathbb{K}$ *such that* $f(a) = 0$.

Example 2.1.4. *Let us consider the affine line* $\mathbb{C}^1$. *Every ideal in* $\mathbb{C}[x]$ *is principal, so every algebraic set is the set of zeros of a single polynomial.*

Since $\mathbb{C}$ *is algebraically closed, every nonzero polynomial* $f(x)$ *can be written* $f(x) = c(x - a_1) \cdots (x - a_n)$ *with* $c, a_1, \ldots, a_n \in \mathbb{C}$.

Then $Z(f) = \{a_1, \ldots, a_n\}$. *Thus the algebraic sets in* $\mathbb{C}^1$ *are the finite subsets, the empty set and the whole space. Thus the Zariski topology on* $\mathbb{C}^1$ *is the topology whose open sets are the empty set,* $\mathbb{C}^1$ *and the complements of finite subsets.*

2.2 Affine Varieties

Definition 2.2.1. *A nonempty subset* Y *of a topological space* X *is irreducible if it cannot be expressed as the union* $Y = Y_1 \cup Y_2$ *of two proper subsets, each one of which is closed in* Y. *The empty set is not considered to be irreducible.*

Example 2.2.1. $\mathbb{C}^1$ *with its Zarisky topology is irreducible, because its only proper closed subsets are finite.*

Definition 2.2.2. *Given any subset* $A \subseteq \mathbb{K}^n$ *we define the ideal of* A, $I(A)$ *in*

$\mathbb{K}[x_1, x_2, ..., x_n]$ *by*

$$I(A) = \{f \in \mathbb{K}[x_1, ..., x_n] : f(P) = 0\}$$

for all $P \in A$.

Ideals $I(A)$ satisfy the following properties:

(a) If A_1, A_2 are subsets of $\mathbb{K}^n$ and $A_1 \subseteq A_2$, then $I(A_2) \subseteq I(A_1)$.

(b) For any two subsets A_1, A_2 of $\mathbb{K}^n$, we have $I(A_1 \cup A_2) = I(A_1) \cap I(A_2)$.

(c) If A is irreducible, then $I(A)$ is prime. Indeed, if $fg \in I(A)$, then

$$A \subseteq Z(fg) = Z(f) \cup Z(g).$$

Thus

$$A = (A \cap Z(f)) \cup (A \cap Z(g)),$$

both closed subsets of A. Since A is irreducible, we have either $A = (A \cap Z(f))$, in which case $A \subseteq Z(f)$, or $A = (A \cap Z(g))$, in which case $A \subseteq Z(g)$. Hence either $f \in I(A)$ or $g \in I(A)$.

The converse of (c) is true too, and is a consequence of the Hilbert's Nullstellensatz (one of the most significant results of algebraic geometry, which translated from the original German, means "Hilbert's theorem of zeros." that we state below.

Theorem 2.2.2. ***Hilbert's Nullstellensatz:*** *Let* $\mathbb{K}$ *be an algebraically closed field. Then all maximal ideals of the polynomial ring* $\mathbb{K}[x_1, ..., x_n]$ *are of the form*

$$I(a) = (x_1 - a_1, ..., x_n - a_n)$$

for some $a = (a_1, ..., a_n) \in \mathbb{K}^n$.

See Algebraic Geometry, Andreas Gathmann [9] for a proof of the Hilbert's Nullstellensatz.

Definition 2.2.3. *An* ***affine variety*** W *is an irreducible closed subset of* $\mathbb{K}^n$.

More technically, an affine variety over some algebraically closed field $\mathbb{K}$ is the zero-locus of a set of polynomials S in $\mathbb{K}[x_1, ..., x_n]$.

In the affine n-space $\mathbb{K}^n$. This is the set of solutions to a polynomial in n variables with coefficients defined in our field $\mathbb{K}$ that generates a prime ideal. This lets us discuss shapes in space in a much more abstract sense.

Example 2.2.3. *Here are some examples of affine varieties:*

1. *Affine n-space itself is an affine variety, since* $\mathbb{A}^n = Z(0)$. *Similarly, the empty set* $\emptyset = Z(1)$ *is an affine variety.*

2. *Any single point in* $\mathbb{A}^n$ *is an affine variety: because we have* $(c_1, ..., c_n) = Z(x_1 - c_1, ..., x_n - c_n)$.

3. *Every linear functional f on V, where V is a n-dimensional vector space over $\mathbb{K}$, can be seen as an element of $\mathbb{K}[x_1, \cdots, x_n]$, and we know that every subspace W of V of dimension m is the intersection of n-m kernels of linear functionals, i.e., $W = \{P \in V | f_i(P) = 0\}$, where f_i are functionals on V and i=1,2,..,n-m. Thus, all linear subspaces are affine varieties.*

The idea of a subvariety is similar to the familiar idea of a subgroup.

Definition 2.2.4. *A* ***subvariety*** *is a subset of a variety that is itself also a variety.*

2.3 Projective Spaces

Definition 2.3.1. *For any vector space V of dimension n over a field $\mathbb{K}$, the* ***projective space*** *of V is denoted by $\mathbb{P}(V)$ (sometimes the field $\mathbb{K}$ replaces V or subscript notation is used instead) which is the set of all 1-dimensional subspaces of V. For details see [7,page 84]*

We will say that the projective space $\mathbb{P}(V)$ has dimesion n-1. A projective space of dimension 1 or 2 we will call a projective line or a projective plane respectively. If $V = \mathbb{K}^{n+1}$ we write $\mathbb{P}^n$ instead of $\mathbb{P}(V)$.

Remark Observe that the projective $n-1$-dimensional space can be written as

$$\mathbb{P}^n = \{(a_0, \ldots, a_n) \in \mathbb{K}^{n+1} \mid a_i \in \mathbb{K} \text{ not all } a_i = 0\}/\sim,$$

where the equivalence relation $\sim$ is defined by

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n),$$

if and only if there is $\lambda \in \mathbb{K}$ such that $a_i = \lambda b_i$ for all i. We write

$$\mathbb{P}^n = \{(a_0 : \ldots : a_n) \mid a_i \in \mathbb{K}\},$$

and we call $(a_0 : \ldots : a_n)$ *homogeneous coordinates* of a point in $\mathbb{P}^n$.

Definition 2.3.2. *An affine non-empty algebraic set $X \subseteq \mathbb{K}^n$ is called a* **cone** *if for all $(a_0, \ldots, a_n) \in X$ and all $\lambda \in \mathbb{K}$ we have $(\lambda a_0, \ldots, \lambda a_n) \in X$. If $X \subseteq \mathbb{P}^n$ is a projective algebraic set then*

$$C(X) = \{(a_0, \ldots, a_n) \in \mathbb{K}^n \mid (a_0 : \ldots : a_n) \in X\} \cup \{0\}$$

is a cone and it is called the cone over X [based on 9, page 39].

Example 2.3.1. *If X is a hyperbola contained in the projective plane $\mathbb{P}(\mathbb{R}^3)$ identified with the plane $Z = 1$ which contains $(0, 0, 1)$ then the cone $C(X)$ is the union of all lines in $\mathbb{R}^3$ which contain 0 and meet this hyperbola. This set is exactly what we intuitively call a cone.*

Homogenous polynomials are polynomials where all terms have the same degree.

Definition 2.3.3. *An ideal I in a ring $\mathbb{K}[x_1, ..., x_n]$ is said to be* ***homogeneous*** *if for each $f \in I$, the homogeneous components f_i of f are in I as well.*

If $f = \sum_i f_i \in I \subset \mathbb{K}[x_1, x_2, ...x_n]$ where f_i are the homogeneous polynomials, then we said that f_i are the homogeneous components of f.

This property is not true in most ideals. For instance, let $I = \langle y - x^2 \rangle \subset \mathbb{K}[x, y]$. The homogeneous components of $f = y - x^2$ are $f_1 = y$ and $f_2 = -x^2$. Neither of these polynomials is in I since neither is a multiple of $y - x^2$. Hence, I is not a homogeneous ideal.

Example 2.3.2. $\mathbb{P}(\mathbb{R}^2)$*: one usually thinks of this by fixing a reference line, for instance the line $y = 1$. Then $\mathbb{P}(\mathbb{R}^2)$ is every 1-dimensional subspace that intersects the reference line plus the one line that is parallel to the reference line.*

Equivalently, $\mathbb{P}(\mathbb{R}^2) = \mathbb{R} \cup \{\infty\}$. Also we can see $\mathbb{P}(\mathbb{R}^2)$ as a semicircle centered in (0,0) without one extreme point, because every 1-dimensional subspace that intersects this semicircle in a point.

The above example shows that $\mathbb{P}(\mathbb{R}^2)$ can be identified with a subset of $\mathbb{R}^2$. In general $\mathbb{P}(V)$ can be identified with a subset of V.

2.4 Projective Varieties

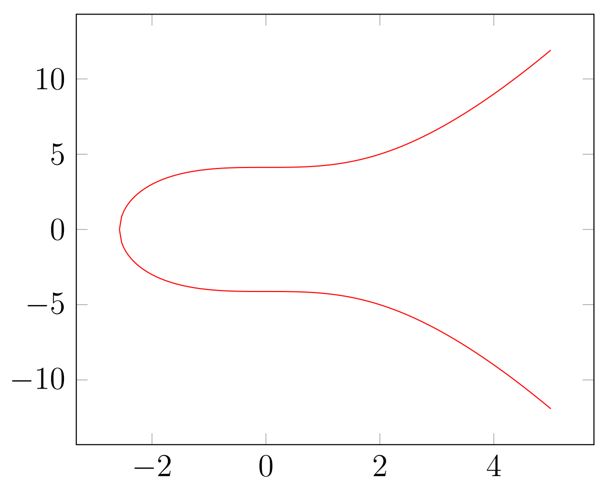
Definition 2.4.1. *A subset $X \subseteq \mathbb{P}^n$ is an* ***algebraic subset*** *of the projective space. if X as subset of $\mathbb{K}^n$ is the zero locus of a homogeneous ideal in $I \subseteq \mathbb{K}[x_0, ..., x_n]$, and X is* ***irreducible*** *if X is algebraic and the corresponding ideal I is prime.*

Definition 2.4.2. *A subset $X \subseteq \mathbb{P}^n$ is a **projective algebraic variety**, if X is an irreducible algebraic subset of the projective space.*

Take an algebraically closed field $\mathbb{K}$, and define a projective n-space $\mathbb{P}^n$ over it. Then, the **projective variety** is a subset of $\mathbb{P}^n$ that is the zero-locus of some finite family of homogenous polynomials.

Example 2.4.1. *An important area of modern mathematical research deals largely with objects called **elliptic curves**, which are non-singular plane curves of the form $y^2 = x^3 + ax + b$. While the image below is necessarily in affine space, the projective curve corresponding to the curve in the graph below is an elliptic curve in $\mathbb{P}^2$.*

Elliptic Curve $y^2 = x^3 + 17$



Definition 2.4.3. *A subset Y of a topological space X is said to be **locally closed** if it is the intersection of an open and a closed subset.*

The following result provides some equivalent definitions:

Proposition 2.4.2. *The following are equivalent:*

1. *Y is locally closed in X.*

2. *Each point in Y has an open neighborhood $U \subseteq X$ such that $U \cap Y$ is closed in U (with the subspace topology).*

3. *Y is open in its closure $\overline{Y}$ (with the subspace topology).*

Definition 2.4.4. *A* ***quasi-projective variety*** *is a locally closed subset of projective n-space $\mathbb{P}^n$.*

As the name implies, a quasi-projective variety is almost a projective variety. A quasi-projective variety is a locally closed subset of a projective variety. While we said affine varieties might be the simplest, they are not the most general. This is because affine varieties are quasi-projective, while there exist locally closed subsets of projective varieties that are not affine, but are quasi-projective.

In classical algebraic geometry all varieties were quasi-projective. Due to Andre Weil though, we now have a concrete definition of varieties that are not quasi-projective.

Example 2.4.3. 1. *Every projective algebraic set X is a quasi-projective variety.*

2. *Every affine algebraic set $X \subseteq \mathbb{K}^n \equiv U_0$ is a quasi-projective variety.*

$$X = X \cap U_0.$$

3. Every open subset of a quasi-projective variety is a quasi projective variety.

4. Every closed subset of a quasi-projective variety is a quasi-projective variety.

If $X \subset \mathbb{K}^n$ and $Y \subset \mathbb{K}^m$ are closed subvarieties, then $X \times Y$ is a closed subvariety of $\mathbb{K}^{n+m} = \mathbb{K}^n \times \mathbb{K}^m$.

Proposition 2.4.4. *A product $X \times Y$ of irreducible varieties is irreducible.*

Proof. Suppose that $X \times Y = Q_1 \cup Q_2$, with each Q_i a closed subset of $X \times Y$. For each $x \in X$, the closed set $x \times Y$ is isomorphic to Y, and is therefore irreducible. Since $x \times Y = ((x \times Y) \cap Q_1) \cup ((x \times Y) \cap Q_2)$ either $x \times Y \subset Q_1$ or else $x \times Y \subset Q_2$.

The subset $X_1 \subset X$ consisting of those $x \in X$ with $x \times Y \subset Q_1$ is a closed subset. We have $X_1 = \bigcap_{y \in Y} X_y$, where X_y is the collection of points $x \in X : x \times y \in Q_1$.

Since $X_y \times y = (X \times y) \cap Q_1, X_y$ and hence X_1 is closed.

Similarly define the closed subset X_2. Since $X = X_1 \cup X_2$ and X is irreducible, we either have $X = X_1$ or $X = X_2$. But $X = X_i$ implies $X \times Y = Q_i$, which proves $X \times Y$ is irreducible. This proof is from [12,page 211].

□

Definition 2.4.5. *Let R be a ring containing a subring A. An element $b \in R$ is **integral** over A if it is root in a monic polynomial with coefficients in A. The set of elements integral over A is a subring called the **integral closure**.*

Chapter 3

Zariski Topology

Here we will begin our discussion of a very important mathematical structure that is very prevalent in algebraic geometry. Certain special cases of varieties and ideals rely on ideas presented in this chapter, and thus did not appear earlier. Instead, they will appear here, so make sure that there is a firm grasp of the ideas presented in the two previous chapters.

3.1 Zariski Topology

Zariski topology took particular importance around 1950. It is named after Oscar Zariski. The Zariski topology is especially good for studying polynomial equations in algebraic geometry. It provides tools of topology for the study of algebraic varieties.

In the previous chapter we defined affine space $\mathbb{K}^n$, and affine algebraic sets as those for which there exists a set $S \subseteq \mathbb{K}[x_1, x_2, \cdots, x_n]$ such that

$$V = Z(S) = \{P \in \mathbb{K}^n | f(P) = 0, \forall f \in S\}.$$

We see that $\mathbb{K}^n = Z(0)$ and the empty set $\emptyset = Z(1)$ are affine algabraic sets. Also we show that, finite unions and arbitrary intersections of affine algebraic sets are affine algebraic sets too.

Proposition 3.1.1. *Let $\mathbb{K}^1$ be a 1-dimensional vector space over $\mathbb{K}$ equipped with the Zariski topology. Then the following hold:*

1. *The closed subsets of $\mathbb{K}^1$ are finite subsets and $\mathbb{K}^1$ itself.*
2. *If $f \in \mathbb{K}[x_1, \cdots, x_n]$, then $f : \mathbb{K}^n \longrightarrow \mathbb{K}$ is continuous.*

Proof.

1. Of course $\mathbb{K}^1 = Z(0)$ and $\emptyset = Z(1)$ are closed. A non-empty finite subset $\{\alpha_1, \cdots, \alpha_m\}$ of $\mathbb{K}^1$ is the zero locus of the polynomial $(x-\alpha_1)\cdots(x-\alpha_m)$, so it is closed.

   Conversely, if S is a subset of $\mathbb{K}[x]$ and $I = \langle S \rangle$ is the ideal generated by S in $\mathbb{K}[x]$, then $Z(S) = Z(I)$, its zero locus is the finite subset of $\mathbb{K}^1$ consisting of the zeros of f. Where f is the generator of I.

2. To show f is continuous, we must show that the inverse image of every closed subset of $\mathbb{K}$ is closed. The inverse image of $\mathbb{K}$ is $\mathbb{K}^n$, which is closed.

   And $f^{-1}(\emptyset) = \emptyset$ is closed. Since the only other closed subsets of $\mathbb{K}$ are the non-empty finite subsets, it suffices to check that $f^{-1}(x)$ is closed for each $x \in \mathbb{K}$.

   But $f^{-1}(x)$ is precisely the zero locus of $f - f(x)$

$\square$

3.2 Zariski Topology on Affine and Projective spaces

Definition 3.2.1. *The Zariski topology on $Spec(R)$ is has closed sets defined by:*

$$V(S) = \{I \in Spec(R) : S \subset I\}$$

where S is an ideal in R.

Example 3.2.1. *Every point $P = (a_1, ..., a_n)$ in $\mathbb{K}^n$ is closed. Because if $f = (x_1 - a_1, ..., x_n - a_n)$, then $P = Z(f)$.*

Definition 3.2.2. *If S is a set of homogeneous polynomials in $\mathbb{K}[X_0, ..., X_n]$, we define its zero set $Z(S) \subset \mathbb{P}^{n-1}$ by*

$$Z(S) = \left\{P \in \mathbb{P}^{n-1} \mid f(P) = 0, \forall f \in S\right\}.$$

If I is a homogeneous ideal of $\mathbb{K}[X_0, \cdots, X_n]$, we define its zero set $Z(I) \subset \mathbb{P}^{n-1}$ to be the zero set of the set of all homogeneous polynomials in I.

Lemma 3.2.1. *Zariski's Lemma: Let k be a field and $k = \mathbb{K}[x_1, \cdots, x_n]$. Then every x_i must be algebraic over $\mathbb{K}$.*

Proof. Suppose x_1 is not algebraic over $\mathbb{K}$, but $(x_2, ..., x_n)$ are algebraic over $\mathbb{K}[x_1]$. Since each x_i is algebraic over $\mathbb{K}[x_1]$ there are polynomials $f_j(x_1)$ (where j goes from 2 to n) such that each x_j is integral over the domain (see definition 2.4.6).

$$A = \mathbb{K}[x_1]\left[\frac{1}{f_2(x_1)}, \dots, \frac{1}{f_n(x_1)}\right]$$

Since $k = \mathbb{K}[x_1][x_2, \cdots, x_n]$. Then k is integral over $\mathbb{A}$. Thus, $\mathbb{A}$ is a field, which means $\mathbb{A} = \mathbb{K}[x_1]$ which is impossible. This concludes the proof.[14]

$\square$

Definition 3.2.3. *The Zariski topology on $\mathbb{P}^n$ is the topology for which the closed sets are the subsets of the form $Z(I)$ for some homogeneous ideal $I \subset \mathbb{K}[x_0, \cdots, x_n]$.*

Definition 3.2.4. *The Zariski topology of an algebraic variety is the topology whose closed sets are the algebraic subsets of the variety.*

3.3 Coordinate Rings

In algebraic geometry, our objects are varieties, and because varieties are defined by polynomials, it will be appropriate to consider polynomial functions on them. Now we will have further facts on varieties. A coordinate ring which is also called ring of polynomial functions, is known as coordinate ring because it is defined on coordinate functions.

Definition 3.3.1. *Let $\mathbb{K}$ be an algebraically closed field. The* **coordinate ring** *of* *an affine variety $V \in \mathbb{K}^n$ is defined by*

$$\mathbb{K}[V] = \mathbb{K}[x_1, x_2, \cdots, x_n] \diagup I(V),$$

where $I(V)$ is the ideal formed by the set of polynomials $f(x_1, \cdots, x_n)$ with coefficients in the field $\mathbb{K}$ which equal zero for all points in the variety V. [13,page 47]

Example 3.3.1. *Consider a variety $V = \{(x, y)|x^2 = y^3\}$ as a subset of the real plane $\mathbb{R}^2$.*

The ring of real-valued polynomial functions defined on V can be identified with the quotient ring $\mathbb{R}[x, y] \diagup (x^2 - y^3)$, and this is the coordinate ring of V.

Proposition 3.3.2. *Given an affine algebraic set $V \subseteq \mathbb{K}^n$. If V is irreducible then $\mathbb{K}(V) \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is prime and $\mathbb{K}[V]$ is a domain.*

Proof. Since $\mathbb{K}[V] \cong \mathbb{K}[x_1, \ldots, x_n] \diagup I(V)$, to prove primality, assume that V is reducible. Then $V = V_1 \cup V_2$ for some $V_i \subsetneq V$. Then $Z(V_i) \supsetneq Z(V)$ and $Z(V) = Z(V_1) \cap Z(V_2)$.

If $f_i \in Z(V_i) - Z(V)$ for $i = 1, 2$, then $f_1 f_2 \in Z(V) \Longrightarrow Z(V)$ is not prime.

That $\mathbb{K}[V]$ is a domain, it is a consequence that if R is a ring and I is a prime ideal of R, then R/I is a domain.

□

Example 3.3.3. *The variety* $Z(x^2 - y^2) \subset \mathbb{R}^2$ *is reducible, because it is the union of* $Z(x + y)$ *and* $Z(x - y)$.

In relation to the quasi projective varieties views in chapter 2 we have:

Definition 3.3.2. *The dimension of an irreducible quasi-projective variety* V *is the transcendence degree of* $\mathbb{K}(V)$ *over* $\mathbb{K}$. *[8,page 15]*

Example 3.3.4. $\mathbb{K}^3 \supset Z(xz, xy) = Z(x) \cup Z(y, z)$ *and so* $\mathbb{K}[y, z] \cong \mathbb{K}[x, y, z] \diagup (x)$ *has transcendence degree 2 and* $\mathbb{K}[x, y, z] \diagup (y, z) \cong \mathbb{K}[x]$ *has transcendence degree 1.*

Bibliography

[1] M. Atiyah, I. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969).

[2] Reid, Miles, *Undergraduate Commutative Algebra*, Cambridge Univ. Press, 1996.

[3] Daniel Berlyne, *translation of the paper "Idealtheorie in Ringbereichen"* by Emmy Noether, Submitted on 11 Jan 2014.

[4] Dilip P. Patil, Uwe Storch *Introduction to Algebraic Geometry and Commutative Algebra*, word scientific, 2010.

[5] Bhubaneswar Mishra, *Algorithmic Algebra*, Springer-Verlag Incorporated,1993.

[6] Larry C. Grove, *Algebra*, Dover Publications, 2004.

[7] Johannes Ueberberg, *Basic Algebraic Geometry 1*, Springer-Verlag, 1999.

[8] Johannes Ueberberg, *Foundations of Incidence Geometry: Projective and Polar Spaces*, Springer-Verlag, 2011.

[9] Andreas Gathmann, *Algebraic Geometry* , University of Kaiserslautern, 2014.

[10] R. Y. Sharp, *Basic Algebraic Geometry 1*, Cambridge University Press, 2000.

[11] Sarges, H., *Ein Beweis des Hilbertschen Basissatzes. J. reine angew.* Math. 283/284 (1976), 436-437.

[12] V.I. Danilov, V.V. Shokurov, *Algebraic Curves, Algebraic Manifolds and Schemes* , Mar 17, 1998.

[13] James Milney, *algebraic geometry* , March 19, 2008.

[14] http://arxiv.org/abs/1506.08376, *math.AC*, Jun 28, 2015.

Vita

Rasha Sulaiman Almohammadi
Taibah University, B.SC., Math, (27 Jun 2005)
Publications: N/A
Conference Presentations: N/A