ISSN 0381-9019

A LOWER BOUND ON THE NUMBER OF ONE-FACTORS IN BICUBIC GRAPHS

J. D. HORTON
SCHOOL OF COMPUTER SCIENCE
UNIVERSITY OF NEW BRUNSWICK
FREDERICTON, NEW BRUNSWICK
CANADA, E3B 5A3

TR84-024, JUNE 1984

A LOWER BOUND ON THE NUMBER OF ONE-FACTORS IN BICUBIC GRAPHS

by

J.D. HORTON

School of Computer Science

University of New Brunswick

Fredericton, New Brunswick

CANADA, E3B 5A3

Technical Report 84-024

A LOWER BOUND ON THE NUMBER OF ONE-FACTORS IN BICUBIC GRAPHS

ABSTRACT

The number of one-factors in a bicubic graph is shown to be more than polynomial in the number of vertices. Thus the permanent of a matrix of 0's and 1's in which each row and column includes precisely three 1's is more than polynomial. This improves the known lower bound of 3n. The form of the bound for a graph with 2n vertices is cn^{an}, where c is some constant and $a < \log_2(9/4)=.85...$.

OTHER KEYWORDS: permanents, 0-1 matrices,

1. Introduction

All graphs considered in this paper are bicubic, that is bipartite and cubic so that they can be assumed to have 2n vertices and 3n edges. Loops are not allowed, but multiple edges are allowed. Brualdi has shown that all such simple graphs for $n \geq 10$ have at least 3n one-factors [3], improving upon a result of Hartfiel and Crosby [4] who showed that bicubic graphs have at least 3n-3 one-factors. In this paper, the number of one-factors is shown to be greater than any given polynomial in n for large enough n.

As pointed out by Brualdi, the number of one-factors in a simple bipartite graph is the permanent of the adjacency matrix of the graph in which the rows correspond to vertices in one block of bipartition, and the columns correspond to vertices in the other block of the bipartition. Thus, a lower bound on the number of one-factors in a bicubic graph with 2n vertices is also a lower bound on the permanent of an nxn matrix of 0's and 1's with three 1's in each row and column.

One should note that the upper bound on the number of one-factors in a bicubic graph with 2n vertices, $6^{n/3}$, that Brualdi derives from Bregman [2], applies only to simple graphs. If one takes a circuit with 2n vertices and doubles every second edge, the graph obtained has 2^n+1 one-factors. As well, this graph has n edges that are only in one 1-factor, which also apparently contradicts Brualdi's bound of n one-factors ($n \geq 10$) that include a given edge. However, Brualdi dealt only with simple graphs, whereas in this paper multiple edges are required to allow induction to proceed.

The approach taken is to get a lower bound for the number of one-factors in a bicubic graph of size 2n in terms of the number of one-factors in smaller graphs. Unfortunately, the recursive formulas found are neither tight nor simple. Indeed, several different cases must be considered.

Define f(G) to be the number of one-factors in G, and f(n) to be the minimum value of f(G) for G a bicubic graph of order 2n. Define $f_1(G,e)$ to be the number of one-factors including the specified edge e in graph G, $f_1(G)$ to be the minimum of $f_1(G,e)$ for any edge e where e is not in a two-edge cut, and $f_1(n)$ to be the minimum value of $f_1(G)$ for G a bicubic graph of order 2n. Define $f_2(G,e)$ to be the number of one-factors excluding a specified edge e, $f_2(G)$ to be the minimum of $f_2(G,e)$ for any edge e in G, and $f_2(n)$ to be the minimum value of $f_2(G)$ for G a bicubic graph of order 2n. Note that $f(G) = f_1(G,e) + f_2(G,e)$; and $f(G) \geq f_1(G) + f_2(G)$. The latter inequality only follows because every bicubic graph G has an edge e that is not in a two-edge cut.

Let x,y, and z be three edges of a graph incident with a node v of a graph G. Then

$$f(G) = f_1(G,x) + f_1(G,y) + f_1(G,z); \text{ and}$$

$$2f(G) = f_2(G,x) + f_2(G,y) + f_2(G,z).$$

The second equation shows that $2f(G) \geq 3f_2(G)$, so $2f(n) \geq 3f_2(n)$. Again, the first equation does not lead directly to general inequalites because x,y, and z could be part of a two-edge cut. However, if G has a vertex that is not the endpoint of an edge in a two-edge cut, i.e. not in a two-vertex cut set, then $f(G) \geq 3f_1(G)$.

In this paper log (x) means the logarithm base 2 of x.

2. Recursive lower bounds for f

Let G be any bicubic graph with 2n vertices, n $\geq$ 2. If G has a multiple edge, then G has a substructure as shown in figure 1(a). The two vertices can be removed, and the whole structure replaced by a single edge to obtain a graph G' as shown in figure 1(b).

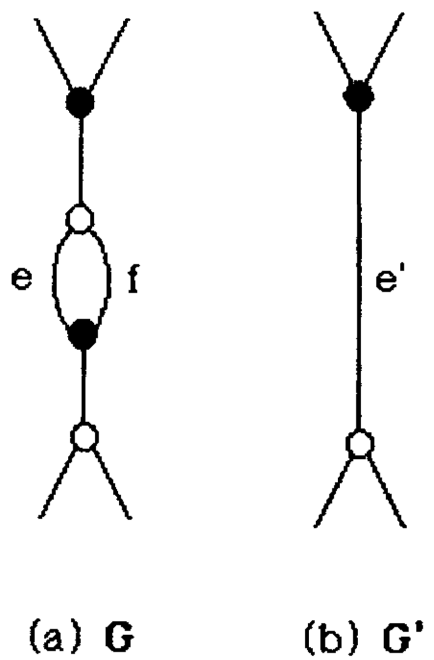


Figure 1. Multiple edge reduction.

Now any one-factor of G must use either one of the double edges, or both of the edges adjacent to the double edge. But a one-factor of G that uses one of the double edges is equivalent to a one-factor of G' that does not use e', while a one-factor that does not use the double edges is equivalent to a one-factor of G' that uses e'. Hence $f(G) = f_2(G,e) + f_1(G,e) = f(G^1) + f_2(G^1,e)$, so

$$f(G) \geq f(n-1) + f_2(n-1). \quad (1)$$

From now on, we assume G has no multiple edges.

Let 2k be the length of the shortest circuit C in G (the girth of G). Then G has at least $2^{k+1}-2$ vertices (see [1], for example). Hence $k \leq \log (n+1)$.

Let the vertices of C be $v_1, v_2, \ldots, v_{2k}$, and let $v_1u_1, v_2u_2, \ldots, v_{2k}u_{2k}$ be edges in G but not in C. If $k > 2$, then the u_i are all distinct. However, the argument still works even if k=2 and the u_i are not distinct. The situation is shown in figure 2(a). Define $e = v_1v_2$. Define G' to be the graph G with $v_1, v_2, \ldots, v_{2k}$ removed, and the edges u_1u_2, $u_3u_4, \ldots, u_{2k-1}u_{2k}$ added. Let $e' = u_1u_2$. Also define G" to be the graph G with the same vertices deleted, but with edges $u_{2k}u_1, u_2u_3, \ldots, u_{2k-2}u_{2k-1}$ added. See figures 2(b) and 2(c).

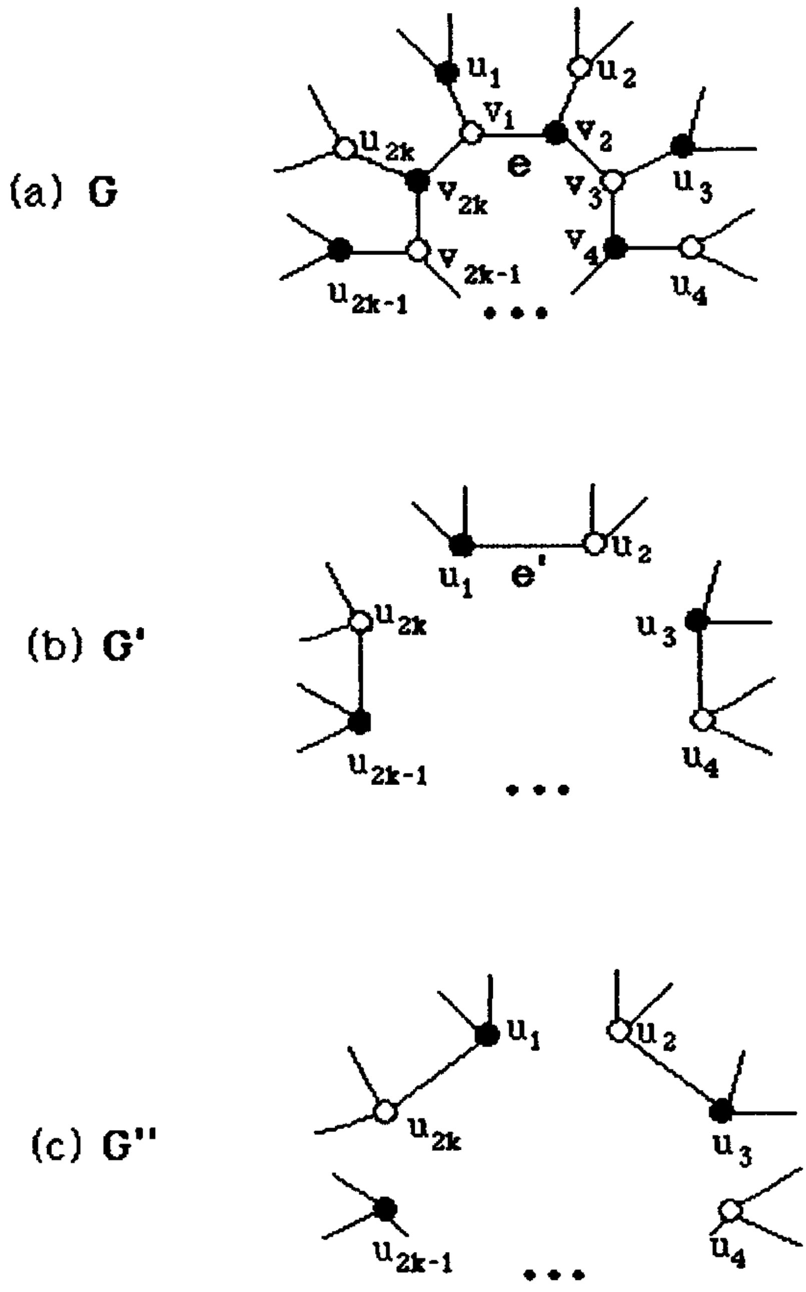


Figure 2. Circuit reduction.

From any one-factor F' of G'', one can make a one-factor F of G as shown in figure 3. If an edge $u_i u_{i+1}$ is in F', put the edges $u_i v_i$, $u_i u_{i+1}$ into F, otherwise put $v_i v_{i+1}$ into F. All other edges of F' are carried directly into F. Note that F cannot contain the edge e. Similarly, one-factors of G' that miss edge e' can be made into one-factors of G that contain e. Hence $f(G) = f_1(G,e) + f_2(G,e) \geq f_2(G',e') + f(G'')$, so

$$f(G) \geq f_2(n-k) + f(n-k), \quad k \leq \log(n+1). \qquad (2)$$

Note that the reduction of a multiple edge is a special case of this shortest circuit reduction, so that the case $k=1$ can be included in this formula. Unfortunately, the bound depends on f_2.

There is one case where this circuit reduction does not work. If G consists solely of two-vertex connected components, then there is no circuit that can be reduced. However, $f(G) = 3^n$, $f_1(G) = 3^{n-1}$, and $f_2(G) = 2(3^{n-1})$, all of which are the maximum possible values for these functions for graphs of a fixed size.

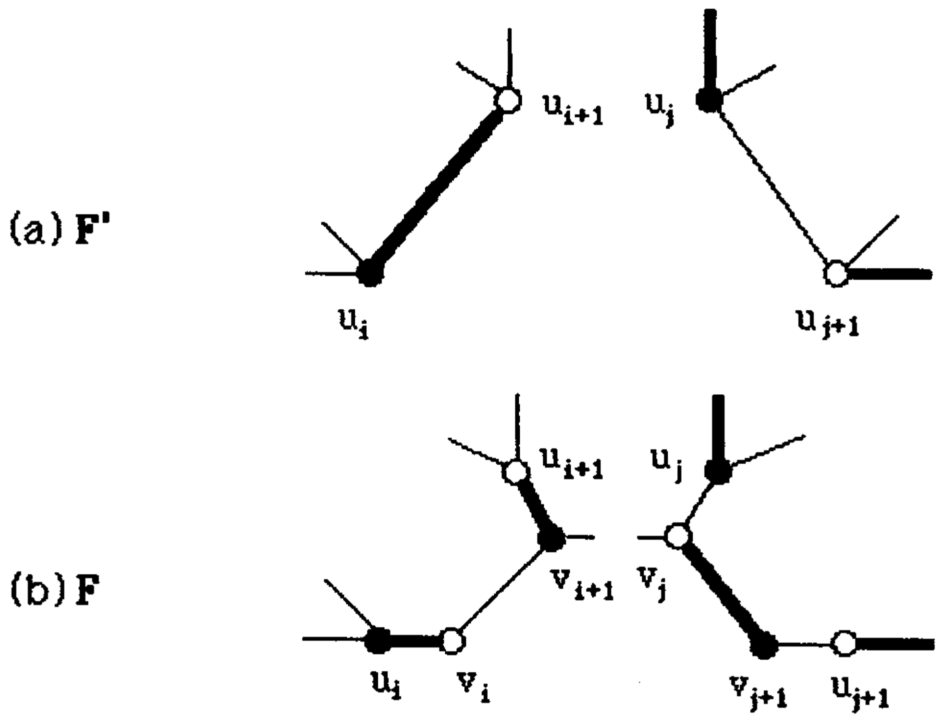


Figure 3. Extending One-factors.

3. Recursive lower bounds for f_2.

Let G be any bicubic graph with 2n vertices and let e be any edge in G. As in section 2, we search for a lower bound on $f_2(G,e)$ to develop a lower bound on $f_2(n)$. If e is a member of a double edge, consider figure 1 again. Again a one-factor of G' that includes the edge e' corresponds to a one-factor of G that contains neither e nor f. Any one-factor of G' that misses the edge v'u' corresponds to a one-factor of G that contains f. Hence $f_2(G,e) \geq f_1(G',e') + f_2(G',e') = f(G')$, so

$$f_2(G,e) \geq f(n-1). \tag{3}$$

We now assume e is not part of a multiple edge.

If e is part of a two-edge cut, then let f be the other cut edge. Given a one factor of G, either both e and f are in it, or neither are in it. Let e = vv' and f = uu' so that v and u are on the same side of the cut. Replace e and f by the edges vu and v'u'. Then the graph falls apart into two graphs, H containing vu, and K containing v'u', as shown in figure 4.

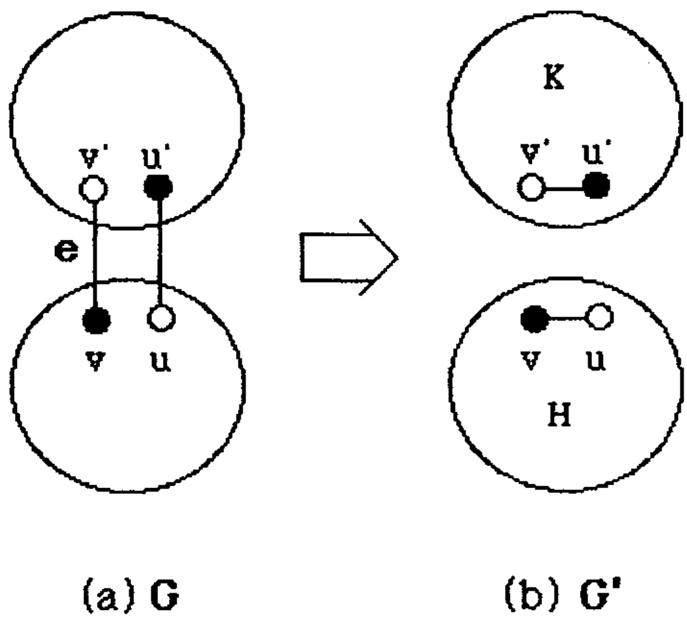


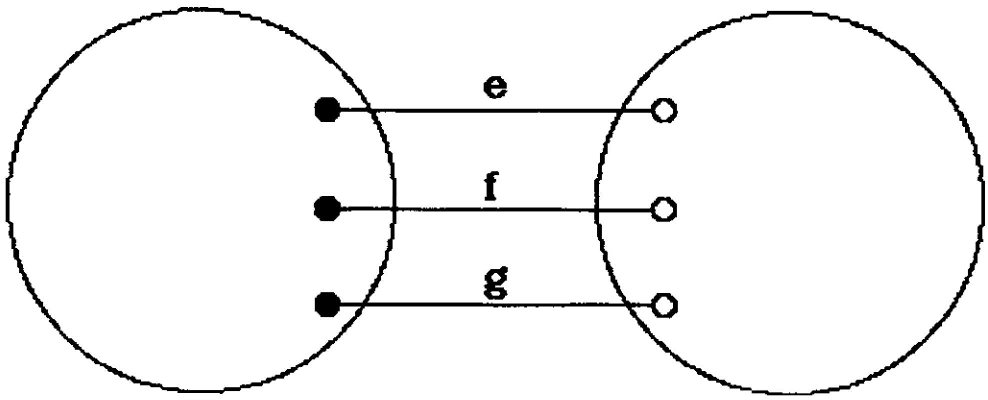
Figure 4. Reducing a two-edge cut.

Thus $f_2(G,e) = f_2(H,vu)f_2(K, v'u')$, so

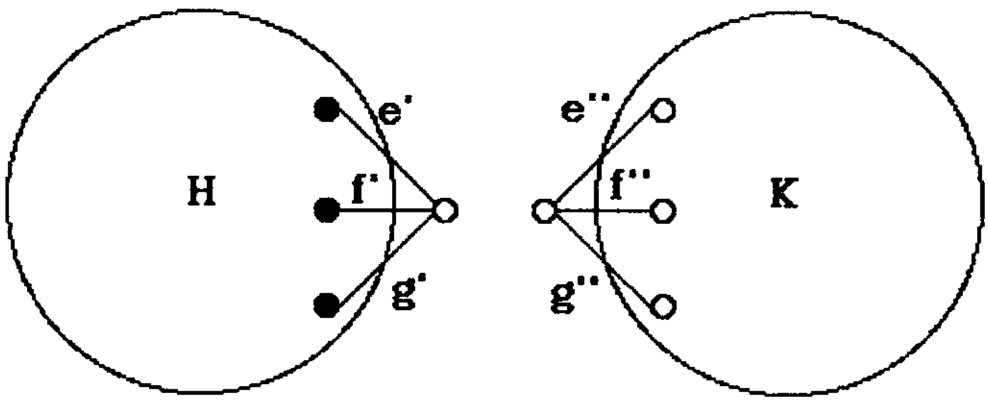
$$f_2(G,e) \geq f_2(k)f_2(n-k) \ , \quad 1 \leq k \leq n-1 \tag{4}$$

Thus we can now assume e is not in a two-edge cut.

If e is in a three-edge cut, {e,f,g}, the situation is depicted in figure 5(a). Cut the three edges, and add two new vertices, one to each component, to form two new bicubic graphs H and K as shown in figure 5(b). In the event that f or g are in two-edge cuts, there is a choice for which edges are to be chosen as part of the three-edge cut. In this case, choose f and g so that the new edges f" and g" are not in two-edge cuts of K.



(a) G



(b) H and K

Figure 5. Reducing a Three-edge Cut

Then $f_2(G,e) = f_1(G,f) + f_1(G,g) = f_1(H,f')f_1(K,f'') + f_1(H,g')f_1(K,g'') \geq f_2(G,e')f_1(k)$. Hence

$$f_2(G,e) \geq f_2(n-k+1)f_1(k), \qquad 3 \leq k \leq (n+1)/2. \tag{5}$$

From now on we can assume e is not part of a three-edge cut, so that neither end of e can be in a set of three vertices whose removal would disconnect G.

Let the edge e be vu, and let 2k be the length of the shortest circuit through e. Let V_i be the set of vertices that are at distance i from vertex v, without using the edge e itself. For $1 < i < 2k-1$, $\circ V_i \circ \geq 3$, since $V_i \cup \{v\}$ is a cut-set of vertices. Also $\circ V_1 \circ = 2$, $\circ V_{2k} \circ \geq 1$, $\circ V_{2k-1} \circ \geq 2$. Hence $2n \geq 3(2k-4)+6$, or $k \leq (n/3)+1$.

The same reduction can be applied to this circuit as was applied in figures 2 and 3. In this case, we find that $f_2(G,e) \geq f(G')$, so that

$$f_2(G,e) \geq f(n-k), \quad 2 \leq k \leq (n/3)+1. \tag{6}$$

Combining inequalities (3) through (6),

$$f_2(n) \geq \min \{f_2(k)f_2(n-k),\ f_2(n-\ell+1)f_1(\ell), f(n-m)\} \tag{7}$$

for $1 \leq k \leq n-1$, $3 \leq \ell \leq (n+1)/2$, $1 \leq m \leq (n/3)+1$.

Unfortunately, the bound depends on f_1 as well as f_2 and f.

4. Recursive lower bounds for f_1.

Let G be a bicubic graph with edge e such that e is not in a two-edge cut. If e is in a three-edge cut, then the reduction in figure 5 can be performed. Then $f_1(G,e) = f_1(H,e')f_1(K,e'')$, so

$$f_1(G,e) \geq f_1(n-k+1)f_1(k) \ , \qquad 3 \leq k \leq n-3. \tag{8}$$

Otherwise, the edge e is in a circuit of length 2k, where $1 \leq k \leq (n/3) +1$. In this case, performing the reduction described in figures 2 and 3 shows that $f_1(G,e) \geq f_2(G',e')$, so

$$f_1(G,e) \geq f_2(n-k), \ 1 \leq k \leq (n/3)+1 \tag{9}$$

As one of (8) and (9) must apply, we have

$$f_1(n) \geq \min \{f_1(n-k+1)f_1(k), \ f_2(n-m)\} \tag{10}$$

for $3 \leq k \leq n-2$, and $1 \leq m \leq (n/3) + 1$. A complete set of recursive lower bounds on f, f_1 and f_2 have now been found.

5. Solving the inequalities

Inequalities (2), (7), and (10) can be used to get lower bounds on f, f_1, and f_2, if we can get bounds on $f(n)$, $f_1(n)$, and $f_2(n)$ for small values of n. These can be found by looking at all bicubic graphs with fewer than 12 vertices. The actual values can be found in Table I.

n	$f_1(n)$	$f_2(n)$	$f(n)$
1	1	2	3
2	2	3	5
3	2	4	6
4	3	6	9
5	4	8	12

Table I: Actual values for $f(n)$, $f_1(n)$, and $f_2(n)$

If one starts to calculate f using these formulas, one soon realizes that inequalities (2) and (6) are used almost exclusively. Other considerations also suggest these are the crucial inequalities, since if conditions (2), (6) and (9) were removed, then an exponential lower bound can be obtained. Inequality (10) concerns f_1 which is not needed in (2) and (6), and so can be ignored.

Combining (2) and (6) to remove f_2, the inequality

$$f(n) \geq f(n-k) + f(n-k-m)$$

is obtained, where $k = [\log(n+1)]$ and $m = 2(n-k)/3 + 1$. The largest values for k and m are chosen to get the "worst" possible case. The asymptotic solution to this recusion relation is

$$f(n) \geq c\ n^{a \log n} \qquad (11)$$

where $a < 1/\log(9/4) = .855$ and c is some constant factor.

Lower bounds on $f_2(n)$ and $f_1(n)$ can now be obtained using (6) and (9). Thus

$$f_2(n) \geq c\ m^{a \log m}$$

where $m = (2n - 3)/3$, and

$$f_1(n) \geq ck^{a \log k}$$

where $k = (4n - 15)/9$. The verification by induction of these formulas for large enough n can now be carried out using the recurrence inequalities, although it is a tedious process.

6. Conclusions

The importance of (11) is not the exact formula, but rather that there is no infinite sequence of bicubic graphs in which the number of one-factors is bounded above by a polynomial. In fact, I suspect that a considerably better lower bound is valid, possibly even an exponential bound. Although most of the recurrence inequalities are reasonably good the critical inequalities (2), (6) and (9) are not tight at all. In fact, I expect that in the reduction of a circuit, these formulas are out by a factor exponential in the length of the circuit, since, a one-factor need not use only "even edges", or only "odd edges" in the circuit as the one-factors counted by these inequalities do.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, MacMillan Press, London, 1976, p. 236.

[2] L.M. Bregman, Some properties of non negative matrices and their permanents, Soviet Math. Dok. 14(1973), 945-979.

[3] R.A. Brualdi, On 1-factors of cubic bipartite graphs, Ars Combinatoria, 9, 1980, pp. 211-219.

[4] D.J. Hartfiel and J.W. Crosby, On the permanent of a certain class of (0,1)-matrices, Canad. Math. Bull. 14(4), 1971, 507-511.

jdh-09

TECHNICAL REPORTS

SCHOOL OF COMPUTER SCIENCE

Number	Date	Author	Title
TR74-001	Feb 1974	L.F. Johnson	A Search Algorithm for the Simple Cycles of a Directed Graph
TR74-002	Oct 1974	W.D. Wasson R. McIssaac	A New Spanning Tree Algorithm
TR74-003	Oct 1974	U.G. Gujar	Remote Job Entry and Output Through APL
TR75-004	Apr 1975	U.G. Gujar	Subroutines with Variable Numbers of Arguments
TR75-005	Jul 1975	L.E. Garey	Block Methods for Nonlinear Volterra Integral Equations
TR75-006	Aug 1975	D.M. Fellows	Comments on "A General Fortran Emulator for IBM 360/370 Random Number Generator 'RANDU'"
TR75-007	Aug 1975	L.E. Garey M. LeBlanc	Quadrature Formulae for Functions of Two Variables and Applications
TR75-008	Sep 1975	L.F. Johnson	Determining Cliques of a Graph
TR75-009	Oct 1975	D.M. Miller	An Algorithm for Determining the Chromatic Number of a Graph
TR76-010	Jan 1976	L.E. Garey	Step by Step Methods for the Numerical Solution of Volterra Integro-Differential Equations
TR76-011	Jan 1976	U.G. Gujar	A device Independent Computer Plotting System
TR76-012	Mar 1976	P.P. Emin	A Partition Monitor for Fast-Batch-Processing with Limited Execution (Fable)
TR77-013	Dec 1976	U.G. Gujar J.A. Fitzgerald	A Driver for Raster-Like Plotting Devices
TR77-014	Jan 1977	U.G. Gujar D.M. Fellows	Automatic Job Scheduling in HASP

Number	Date	Author	Title
TR79-015	May 1979	U.G. Gujar J.M. DeDourek M.E. McIntyre	A Method for Designing a Lexical Analyzer
TR79-016	Jun 1979	U.G. Gujar	ϕSCUBA A Buffered Core Graphics System
TR79-017	Jun 1979	T.A. Middleton	A Transformation Approach to Implementing Aggregate Operations
TR79-018	Jul 1979	T.A. Middleton	On Assignment Between Data Paths
TR79-019	Aug 1979	T.A. Middleton	On the Use of "Fixed" Data Paths for Assignment
TR79-020	Aug 1979	T.A. Middleton	Representing Data Paths by Program Structures
TR79-021	Sep 1982	J.D. Horton	Sets With No Empty Convex 7-Gons
TR83-022	Jun 1983	J.D. Horton	Resolvable Path Designs
TR83-023	Aug 1983	U.G. Gujar F.W.L. So	Computer Display of Characters
TR84-024	Jun 1984	J.D. Horton	A Lower Bound on the Number of One-Factors in Bicubic Graphs