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Abstract

Erdös has defined g(n) as the smallest integer such that any set of g(n) points in the plane, no three collinear, contains the vertex set of a convex n-gon whose interior contains no point of this set. Arbitrarily large sets containing no empty convex 7-gon are constructed, showing that g(n) does not exist for $n \geq 7$. Whether g(6) exists is unknown.

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Esther Klein raised the following combinatorial geometry problem [5]. For $n \geq 3$, let $f(n)$ be the smallest integer such that for any set of $f(n)$ points in the plane, no three collinear, contains the vertex set of a convex n-gon. Determine $f(n)$. It is easy to show that $f(3)=3$ and $f(4)=5$. That $f(5)=9$ was proved in [4]. Erdos and Szekeres determined that $2^{n-2}+1 \leq f(n) \leq \binom{2n-4}{n-2} +1$ [1],[2].

Erdős has raised a similar question. For $n \geq 3$, define $g(n)$ to be the smallest integer such that any set of $g(n)$ points in the plane, no three collinear, contains the vertex set of a convex n-gon whose interior contains no point of the set. We call a n-gon, with no points of the set in its interior, <u>empty</u>. Again, $g(3)=3$ and $g(4)=5$. Harborth has proved that $g(5)=10$ [3]. However, it is not known whether $g(6)$ exists. The main result of this note is that $g(7)$, and hence $g(n)$ for all $n \geq 7$, does not exist.

We construct, for any k, a set of 2^k points with no empty convex 7-gon. Let $a_1a_2 \ldots a_k$ be the binary expansion of the integer $i, 0 \leq i < 2^k$. Note that leading 0's are not omitted. Let $c=2^k+1$, and define $d(i)=\Sigma\, a_j\, c^{j-1}$, summing from $j=1$ to $j=k$. Let p_i be the point $(i, d(i))$, and define S_k to be the set of points $\{p_i | i=0,1,\ldots,2^k-1\}$. Observations:

(a) $\{p_i | i<2^{k-1}\}$ = the left half of S_k = L.

(b) $\{p_i | i\geq 2^{k-1}\}$ = the right half of S_k = R, which is a translate of L.

(c) $\{p_i | i \text{ is even}\}$ = the bottom half of S_k = B.

(d) $\{p_i | i \text{ is odd}\}$ = the top half of S_k = T, which is a translate of B.

(e) L,R,B, and T are all scaled translates of each other. For example, halving the first coordinate while multiplying the second coordinate by c, takes B onto L.

(f) The 180° rotation of the plane about $((2^k-1)/2, \Sigma c^i/2)$ takes T onto B.

(g) All points of T are above any line joining two points of B. The value of c was chosen large enough to make this true. Similarly, all points of B are below any line joining two points of T.

(h) If i and j both have the same last x digits in their binary expansions, and h has a different sequence of x rightmost digits, then whether p_h is above or below the line joining p_i and p_j is determined by the sequences of the last x digits.

Consider any empty convex n-gon A in S_k. We may assume A is contained entirely in neither T nor B. Otherwise if A is contained in B, apply the linear transformation that takes B onto L. A will be transformed into any empty convex n-gon in L. Similarly, if A is contained in T, apply the linear transformation that takes T onto L. Repeat this procedure until a transformed image of A meets both T and B.

Next, consider how many points of A can be in B. Assume p_i and p_j are in $A \cap B$. By (g) above, no point p_h of B, with $i<h<j$, can be above the line segment joining p_i and p_j, since otherwise no point of T could be in A. As well, I claim that $d(h)<d(i)$ and $d(h)<d(j)$. Since p_h is below the line joining p_i and p_j, clearly one of these statements is true. Assume $d(h)<d(i)$, but $d(h)>d(j)$. Let x be the position of the right-most digit at which h and i differ in their binary expansions; let y be the position of the right-most digit at which h and j differ. In both cases, the number with the larger functional value must have a 1 in the position, and the other number a 0. If $x < y$ then p_j must be below the line joining p_i and p_h, by observation (h).

But then p_h is above the line joining p_i and p_j, a contradiction. Hence we can assume that $y < x$. In this case, consider $\ell = j - 2^{k-x}$. The right-most position in which the binary expansions of ℓ and j differ is x, where ℓ has a 1 and j has a 0. On the other hand, ℓ and i must agree in the last k-x positions. By observation (h), p_j is below the line joining p_i and p_ℓ. But since $j-i > j-h \geq 2^{k-y} > 2^{k-x} = j-\ell$, $i < \ell < j$. Then p_ℓ must be both above and below the line joining p_i and p_j, a contradiction. Similarly, $d(j) < d(h) < d(i)$ leads to a contradiction. Therefore $d(h) < d(i)$ and $d(h) < d(j)$.

If $A \cap B$ contained four points $i < h < \ell < j$, then $d(h) < d(\ell)$ and $d(\ell) < d(h)$. Hence $A \cap B$ cannot contain more than three points. By observation (f) above, $A \cap T$ cannot contain more than three points either. Hence A has no more than 6 points.

Whether g(6) exists is still unknown. However, I can give some indications that g(6) does exist.

Lemma 1. Assume S is a set of n points with no empty convex hexagon, $n \geq 5$. Then at most $[(n+11)/3]$ of the points are in the convex hull, $n \neq 8$.

Proof: Let S have x points in its convex hull, (exterior points) and y points in its interior (interior points).

If $y \geq 2$, consider any two points p and q on the convex hull of the set of interior points. There are at most 3 exterior points on the side of the line joining p and q away from the interior points of S. Throwing these points away, we get a set with at most y-2 interior points, and at least x-1 exterior points. This construction yields the induction step required to prove the lemma.

Clearly if y=0, then $x \leq 5$. If y=1, then we can only show that $x \leq 7$, the exception mentioned in the lemma. If y=2, then $x \leq 6$, using the induction step. If y=3, then x<7, as is shown below, which completes the basis for the induction.

Let p_1, p_2, p_3 be the three interior points of a set S of points with no empty convex hexagon. The line joining p_i and p_j has at most 3 exterior points on the side away from the third interior point, on the "outside", as noted above. But also, given two such lines, there are at most two exterior points on the "inside" of both lines. Otherwise the three exterior points in the intersection of the two "insides", together with p_1, p_2 and p_3, would form an empty convex hexagon. Summing the number of exterior points in the three "outsides", and in the intersection of the three pairs of "insides", we can get at most 15. But each exterior point must be counted twice. Therefore $x \leq 7$.

A planar map is said to be <u>cubic</u> if all vertices are of degree 3; a planar map is said to be <u>convex</u> if all interior faces are convex polygons.

<u>Proposition</u>: If S is the vertex set of a cubic convex planar map with 54 or more vertices, then S contains an empty convex hexagon.

<u>Proof</u>: Let the map have n vertices, f faces, and e edges. Obviously, 3n=2e, and Euler's formula applies, so f+n=e+2. Then f=(n/2)+2.

The sum, over all faces, of the number of edges in each face, is 3n. We may assume that the outer face has at most (n+11)/3 edges, by the lemma. Then the average interior face has (3n-(n+11)/3)/((n/2)+1) edges. The value of this expression is greater than 5 if n>52.

However, not any set S can be represented as the vertex set of a cubic convex planar map. Any set with an odd number of points is a simple counterexample. For a more complicated example, consider 2n points at the corners of two regular n-gons, with one inside the other. If the inner n-gon's vertices are close enough to the middle of the outer n-gon's edges, all the n outer vertices must have degree 4 to make a convex map.

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