What Is a Structural Representation?

Lev Goldfarb, Oleg Golubitsky, Dmitry Korkin

Faculty of Computer Science
University of New Brunswick
Canada

October 25, 2001

Abstract

We outline a formal foundation for a "structural" (or "symbolic") object/event representation, the necessity of which is acutely felt in all sciences, including mathematics and computer science. The proposed foundation incorporates two hypotheses: 1) the object's formative history must be an integral part of the object representation and 2) the process of object construction is irreversible, i.e. the "trajectory" of the object's formative evolution does not intersect itself. The last hypothesis is equivalent to the generalized axiom of (structural) induction. Some of the main difficulties associated with the transition from the classical numeric to the structural representations appear to be related precisely to the development of a formal framework satisfying these two hypotheses.

The concept of (inductive) class representation—which has inspired the development of this approach to structural representation—differs fundamentally from the known concepts of class. In the proposed, evolving transformations system (ETS), model, the class is defined by the transformation system—a finite set of weighted transformations acting on the class progenitor—and the generation of the class elements is associated with the corresponding generative process which also induces the class typicality measure.

Moreover, in the ETS model, a fundamental role of the object's class in the object's representation is clarified: the representation of an object must include the class.

From the point of view of ETS model, the classical discrete representations, e.g. strings and graphs, appear now as *incomplete* special cases, the proper completion of which should incorporate the corresponding formative histories, i.e. those of the corresponding strings or graphs.

Concepts which have proved useful for ordinary things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labeled as "conceptual necessities", *a priori* situations, etc. The road of scientific progress is frequently blocked for long periods by such errors.

A. Einstein

1 Introduction

In this paper, a vision of the concept of structural representation is outlined. On the one hand, although the overwhelming importance of structural, or symbolic, representations in all sciences has become increasingly clear during the second half of the 20th century, there were hardly any systematic attempts to address this topic at a fundamental level [1](and, from the point of view of computer science, this is particularly puzzling in view of the central role played by the "data structures" and "abstract data types" in computer science). On the other hand, it is not that difficult to understand the main reasons behind this situation. It appears that there are two related formidable obstacles to be cleared: 1) the choice of the central "intelligent" process, the structure and requirements of which would both drive and justify the choice of the particular form of structural representation, and 2) the lack of any *fundamental* mathematical model whose roots are not related to or motivated by the numeric models. Unfortunately, the role of the latter obstacle is usually completely underestimated. Why is it so? This is explained by the fact that during the mankind's scientific history, so far, we have dealt, basically, only with the numeric models and, during the last two centuries, with their derivatives. The latter should not be really surprising if we look carefully at the vast prehistory of science in general and of mathematics in particular [1],[2]. The new level of mathematical abstractions and the excessive overspecialization (with the resulting narrowing of the view) during the second half of the twentieth century have also contributed to this lack of understanding of the extent of our dependence on the numeric models. [2] What has begun to facilitate this understanding, however, is the emergence of computer "science" in general and artificial intelligence and pattern recognition (PR) in particular. (Although, for political reasons, during the last 20 years, there appeared several very closely related to PR areas, such as machine learning, neural networks, etc, we will refer to all of them collectively by the name of the original area, i.e. pattern recognition, or, occasionally, as inductive learning.)

In light of the above, it is not reasonable to expect to see the transition from the numerically motivated forms of representation, which have a millenia old tradition behind them, to the structural forms of representation to be accomplished in one or several papers. At the same time, one should not try to justify, as it is often done in artificial intelligence, a practically non-existing progress toward that goal by the complexity of the task.

In this work, a fundamentally new formalism, which is a culmination of the research program originally directed towards the development of a unified framework for pattern recognition [4]–[11], is outlined. On the formal side, we have chosen the path of a far-reaching generalization of the Peano axiomatization of natural numbers,[3] the axiomatics that lies at, as well as forms, the very foundation of the present construction of mathematics. With respect to the latter choice, in some

[1] The Chomsky's formal grammar model will be discussed later in the Introduction and in the paper.
[2] There are, of course, rare exceptions (see, for example, [3]).
[3] See, for example, [12] or [13].

sense, there is no other reasonable way to proceed.[4]

In view of the fact that the above newer, more fashionable "reincarnation" of PR have missed probably the most important development within PR during 1960's and 1970's, we very briefly mention this issue, which actually motivated the original development of the proposed model. During these two decades, it gradually became clear to a number of leading researchers in PR that the two basic approaches to PR—the classical vector-space-based approach and the syntactic, or formal-grammar-based approach—each possessing the desirable features lacking in the other should be unified [14]:

> Thus the controversy between geometric and structural approaches for problem of pattern recognition seems to me historically inevitable, but temporary. There are problems to which the geometric approach is ... suited. Also there are some well known problems which, through solvable by the geometric method, are more easily solvable be the structural approach. But any difficult problems require a combination of these approaches, and methods are gradually crystallizing to combining them; the structural approach is the means of construction of a convenient space; the geometric is the partitioning in it.

And although the original expectations for an impending unification were quite high, it turned out that such hopes were naive, not so much with respect to timeliness but with respect to the underestimated novelty of the unified formalism: there was no formal framework that could naturally accommodate the unification. It is interesting to note that the researchers working in the various "reincarnations" of PR are only now becoming aware of the need for, and of the difficulties associated with, the above unification. The enormous number of conferences, workshops, and sessions devoted to so-called "hybrid" approaches attest to the rediscovery of the need for the above unification.

With respect to the above two formidable obstacles, for us, the choice of the central intelligent process reduced to the pattern recognition, or, more accurately, pattern (or inductive) learning process,[5] with the emphasis on the inductive class representation. On the other hand, the development of the appropriate mathematical formalism, not surprisingly, has been and will continue to be a major undertaking (influenced by non-formal considerations coming from biology, chemistry, and astrophysics).

What are some of the main difficulties that we have encountered? In a somewhat historical order, they are as follows. On which foundation should the unification of the above two basic approaches to PR be approached? How do we formalize the concept of inductive class representation? How should the Chomsky's concept of generativity be revised? How do we generalize the Peano axiomatic construction (of natural numbers) to the construction of structural objects? In other words, how do we formally capture the more general inductive (or evolutionary) process of object construction? What is the connection between the class description and the process that generates the class objects? How do we naturally integrate the structural and the metric information in a chosen formalism? And finally, how does an object representation is connected to its class representation and, moreover, how does the object representation changes during the learning process? It is understood that all of this must be accomplished *naturally* within a single general model.

At present, we strongly believe that the concept of "structural" object representation cannot be

[4]As a well-known 19th century German mathematician L. Kronecker mentioned in an after-dinner speech, "God made the integers; all the rest is the work of man".

[5]Inductive learning processes have been marked as the central intelligent processes by a number of great philosophers and psychologists over the last several centuries (see, for example, [15]–[17]).

divorced from that of "evolutionary" object representation, i.e. of the representation capturing the formative object history, and herein lies the fundamental difference between the classical numeric and the structural, or symbolic, representations. In view of this, the currently widely used non-numeric "representations", such as strings, trees, and graphs, in our opinion, cannot be considered as such: since such representations do not record how the corresponding object was constructed, or generated, there is hardly any connection between the object and the class of objects with respect to which, in the current context, the object is to be represented (and interpreted). Moreover, the framework of formal grammars proposed by Chomsky in the 1950's for generating syntactically valid sentences in a natural language does not address these concerns, which is not quite surprising in view of his repeatedly articulated opinion about the essential irrelevance of the inductive learning process to cognitive science (see for example [18], [19]).

In light of the above monumental difficulties related to the development of a formal model for structural representation, the best we can hope for as a result of the present attempt is to propose and outline some key features of such a formalism. We intend to use the proposed outline only as a guide that will be modified in the course of the extensive experimental work in cheminformatics, molecular phylogenetics, data mining, pattern recognition, computer science, and other application areas. At the same time, as always is the case in science, in our immediate experimental and theoretical work, we will be guided by a reasonable interpretation of the current outline.

As far as the state of our main area of research is concerned, we are encouraged by the fact that some other researchers in the area also strongly feel about its unsatisfactory state [20]:

> To summarize, we are missing [in inductive learning]
>
> a. any kind of consensus on what framework to use;
>
> b. any kind of consensus on where we want to go and what issues we want to answer;
>
> c. any kind of consensus on what new approaches to the issue of assumptions we should be investigating.
>
> For supervised learning to become a science, these missing consensus' must be found.

Ultimately, what should make or break the model outlined in this paper—evolving transformations system (ETS)—as an inductive learning model? Since it is the first inductive model postulating explicitly a new form of class representation (which is not borrowed from any existing mathematical formalism), this postulate for inductive learning, as is the case in all natural sciences, can now be experimentally verified. The latter is not possible to do for any other of the current inductive learning models, since they are not insisting on any form of (inductive) class representation, but simply adopting the existing mathematical formalisms, mainly vector-space-based, to the learning problem. Thus, the immediate value of the ETS model is that it is the first formalism developed specifically to address the needs of the inductive learning process; and, moreover, what is more, the model's postulates, first, explain the nature of this central intelligent process, and, second, can now be experimentally verified. In this respect, it is very important to keep in mind the accumulated scientific wisdom regarding the main value of a scientific model: "Apart from prediction and control the main purpose of science is perhaps explanation… " [21] and "Whatever else science is used for, it is explanation that remains its central aim" [22]. The current inductive learning models explain essentially nothing about the nature of this, quite possibly the central, intelligent process.

The paper, including Introduction, is divided into five sections. The second section is devoted to the explication of the underlying, fundamentally new, mathematical structure—inductive structure. The third section introduces the central concepts of the transformation system, the generating process, the class and its typicality measure, as well as presents several simple illustrative examples. The fourth section, although very short, outlines in a very sketchy format some important issues related to the process of inductive learning within the ETS model. It might be useful to note that the size of the fourth section is not proportional to the concentration of ideas in it. In the last section, we mention some of the larger interesting topics for future research.

Finally, a relatively large number of definitions in the paper is easily explained by the absolute novelty of the underlying model and of all the basic concepts. In order to maintain a reasonable length of the paper, the presence of proofs is tapering off as we progress. On first reading, the proofs might be omitted.

2 The inductive structure

2.1 Initial definitions: primitives and composites

We will use the following notation $(m, n \in \mathbb{Z}_+)$

$$\{m, m+1, \dots, n\} = [m, n].$$

If $m > n$, then $[m, n]$ denotes the empty set. As always, for $B \subseteq X$ and a mapping $f : X \to Y$, we denote by $f\big|_B$ the restriction of f to B, and for another mapping $g : Y \to Z$, $g \circ f$ denotes the composition of mappings. Note that we will not distinguish between $f : X \to Y$ and $f : X \to f(X)$. As always, id_A denote the identity mapping of A onto itself.

Definition 1. Let Π be a finite set whose elements are called primitive types, or simply **primtypes**. Moreover, for every $\pi \in \Pi$ two sets

$$\mathrm{init}(\pi) \quad \text{and} \quad \mathrm{term}(\pi)\,,$$

where both sets are subsets of a fixed set A, are given. These sets specify the sets of **initial** and **terminal a-sites**, or abstract sites, for the primtype π. ►

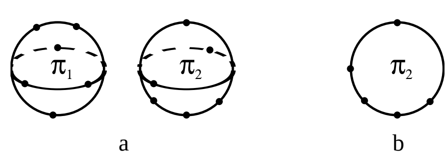

Figure 1: (a) Pictorially, it is convenient to represent primtypes as spheres with the initial a-sites as marked points on its upper part, and the terminal a-sites as marked points on its lower part. Thus, the sites in $\mathrm{init}(\pi) \cap \mathrm{term}(\pi)$ are points on the equator. (b) To simplify subsequent drawings, we replace the spheres by circles and the equator reduces to two, left and right, points.

For every primtype π we introduce the **set of all a-sites**

$$\mathrm{sites}(\pi) = \mathrm{init}(\pi) \cup \mathrm{term}(\pi)\ .$$

Let $\mathcal{S}$ be a fixed countably infinite set whose elements are called *concrete sites*, or simply **sites**. The following definition should be viewed as a far-reaching generalization of the Peano (inductive) definition, or construction, of natural numbers ([12] or [13]).

Definition 2. The set Γ of **composites** is defined inductively[6] as follows. For each $\gamma \in \Gamma$, three subsets of $\mathcal{S}$— $\mathrm{init}(\gamma)$, $\mathrm{term}(\gamma)$ and $\mathrm{sites}(\gamma)$,[7] called the sets of **initial**, **terminal**, and **all sites** of the composite γ —will be constructed inductively.

- λ is the **null** composite whose sets of sites are

$$\mathrm{init}(\lambda) = \mathrm{term}(\lambda) = \mathrm{sites}(\lambda) = \varnothing.$$

- For $\pi \in \Pi$ and a fixed injective mapping

$$f : \mathrm{sites}(\pi) \to \mathcal{S}$$

(called the **site realization** for primtype π) the expression

$$\pi\langle f \rangle$$

signifies the primitive composite, or simply **primitive**[8], whose sets of (concrete) sites are constructed as follows

$$\mathrm{init}(\pi\langle f \rangle) = f(\mathrm{init}(\pi)) \tag{1}$$
$$\mathrm{term}(\pi\langle f \rangle) = f(\mathrm{term}(\pi)) \tag{2}$$
$$\mathrm{sites}(\pi\langle f \rangle) = f(\mathrm{sites}(\pi)) \, . \tag{3}$$

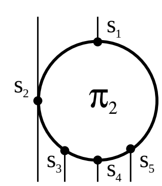

Figure 2: Pictorial representation of primitive $\pi_2\langle f \rangle$ corresponding to the primtype in Fig. 1b.

- For $\gamma \in \Gamma$, $\gamma \neq \lambda$ and $\pi\langle f \rangle \in \Gamma$ satisfying [9]

$$\mathrm{sites}(\gamma) \cap \mathrm{sites}(\pi\langle f \rangle) = \mathrm{term}(\gamma) \cap \mathrm{init}(\pi\langle f \rangle) \, , \tag{4}$$

the expression

$$\gamma \lhd \pi\langle f \rangle$$

[6]See the discussion after this definition.

[7]We use the same notation as that used in Def. 1, since these sets play a similar role.

[8]The difference between primtype π and primitive $\pi\langle f \rangle$ can be compared, for example, to the difference between an element in the periodic table and the same element in a particular compound.

[9]This condition insures the correctness of (5)-(7) below.

signifies the composite γ', whose sets of (concrete) sites are constructed as follows

$$\text{init}(\gamma') = \text{init}(\gamma) \cup [\text{init}(\pi\langle f\rangle) \setminus \text{term}(\gamma)] \tag{5}$$

$$\text{term}(\gamma') = [\text{term}(\gamma) \setminus \text{init}(\pi\langle f\rangle)] \cup \text{term}(\pi\langle f\rangle) \tag{6}$$

$$\text{sites}(\gamma') = \text{sites}(\gamma) \cup \text{sites}(\pi\langle f\rangle). \tag{7}$$

We will call γ' the composite **obtained from** γ **by attachment of primitive** $\pi\langle f\rangle$, where the "attachment" means attaching to each other the identical sites in $\text{term}(\gamma)$ and $\text{init}(\pi\langle f\rangle)$.

Thus, every composite γ is specified by the following **inductive expression** encapsulating its construction process

$$\gamma = \pi_1\langle f_1\rangle \triangleleft \pi_2\langle f_2\rangle \triangleleft \ldots \triangleleft \pi_n\langle f_n\rangle.$$

(See Fig. 3, 4, 5.) We will assume that the above expression is valid for $n = 0$ and in this case denotes λ. ▶

For a composite γ, the set

$$\text{cont}(\gamma) = \text{init}(\gamma) \cap \text{term}(\gamma) \tag{8}$$

will be called the set of **continuation** sites.

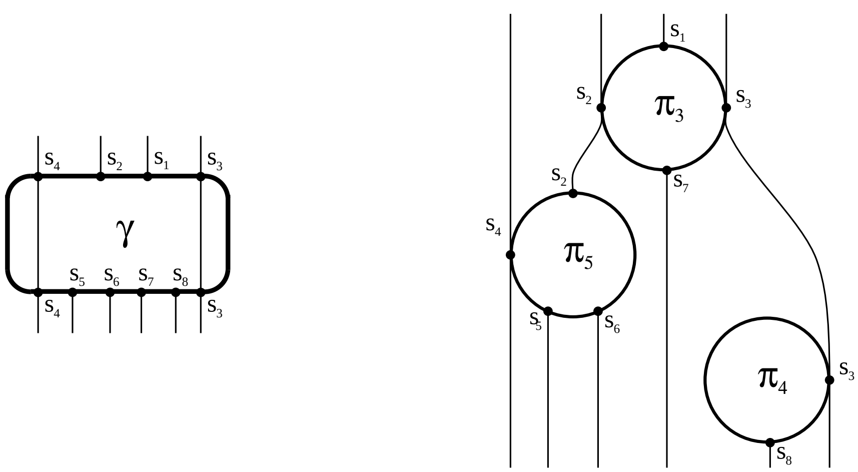

Figure 3: A simplified (left) and a detailed pictorial representation of composite $\gamma = \pi_3\langle f_3\rangle \triangleleft \pi_5\langle f_5\rangle \triangleleft \pi_4\langle f_4\rangle$ with two continuation sites.

It is important to note that, conceptually, we view the set Γ not as derivative with respect to the set of natural numbers, but rather as a more basic set. In particular, we view the set of natural numbers as a very special case of Γ. Consequently, to achieve a necessary degree of rigor, we should rely on the appropriate generalization of Peano axioms (for natural numbers [12]), including the generalization of the induction axiom (see Def. 19). In this paper, however, to retain both a *reasonable* degree of rigor and *accessibility* of the exposition, we will adapt the following inductive schemes.

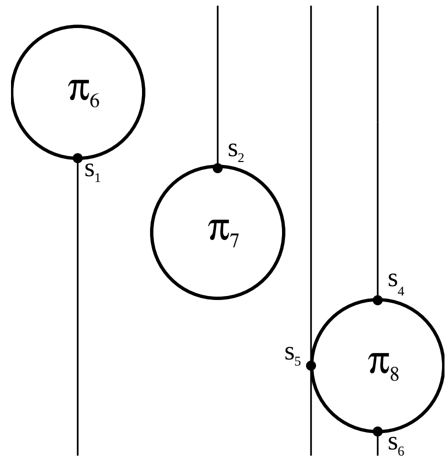

Figure 4: Pictorial representation of a "decoupled" composite $\beta = \pi_6\langle f_6\rangle \lhd \pi_7\langle f_7\rangle \lhd \pi_8\langle f_8\rangle$.

Under the **inductive proof** of a statement $A(\gamma)$ about composites, or **proof by induction on** γ, we mean the following proof scheme.

- Prove that $A(\lambda)$ is true.
- For all $\pi\langle f\rangle$ prove that $A(\pi\langle f\rangle)$ is true.
- If $A(\alpha)$ is true and $\gamma = \alpha \lhd \pi\langle f\rangle$, prove that $A(\gamma)$ is true.

Under the **inductive definition**, or **construction**, of objects $B(\gamma)$ for composites $\gamma \in \Gamma$, we mean the following definition (construction) scheme.

- Construct $B(\lambda)$.
- For any $\pi\langle f\rangle$ construct $B(\pi\langle f\rangle)$.
- Assume that $B(\alpha)$ has been constructed and that $\gamma = \alpha \lhd \pi\langle f\rangle$, then construct $B(\gamma)$.

Note that for a composite γ, the union of its initial and terminal sites could now be *smaller* than the set of all sites. Therefore, for a composite γ, it will also be useful to define the set of its **external** and **internal sites**

$$\begin{aligned} \mathrm{ext}(\gamma) &= \mathrm{init}(\gamma) \cup \mathrm{term}(\gamma) \\ \mathrm{int}(\gamma) &= \mathrm{sites}(\gamma) \setminus \mathrm{ext}(\gamma)\ . \end{aligned} \tag{9}$$

It is quite possible that the fuzzy boundary between the "living" and "non-living" objects can be understood in terms of the assymptotic growth rate of the size of set $\mathrm{ext}(\gamma)$ during the (generative) process of object "development" [10]: normally, the faster the growth rate of $|\,\mathrm{ext}(\gamma)|$ during the "generation" of object γ, the more complex γ is.

[10]See section 3.6.

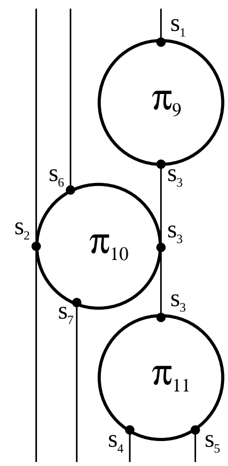

Figure 5: Pictorial representation of composite $\gamma = \pi_9\langle f_9\rangle \lhd \pi_{10}\langle f_{10}\rangle \lhd \pi_{11}\langle f_{11}\rangle$ with one continuation (s_2) and one internal (s_3) sites.

Definition 3. For $\gamma \in \Gamma$ and any injective mapping

$$h : \mathrm{sites}(\gamma) \to \mathcal{S} \ ,$$

called **site replacement**, the composite $\gamma\langle h\rangle$ is defined inductively as follows.

- $\lambda\langle h\rangle \stackrel{\text{def}}{=} \lambda$
- If $\gamma = \pi\langle f\rangle$, then $\gamma\langle h\rangle \stackrel{\text{def}}{=} \pi\langle g\rangle$, where $g = h \circ f$.
- Assume that $\alpha\langle h'\rangle$ has been constructed for any site replacement $h' : \mathrm{sites}(\alpha) \to \mathcal{S}$ and $\gamma = \alpha \lhd \pi\langle f\rangle$, then
$$\gamma\langle h\rangle \stackrel{\text{def}}{=} \alpha\langle h'\rangle \lhd \pi\langle g\rangle \ ,$$
where $h' = h\big|_{\mathrm{sites}(\alpha)}$ and g is as above.

►

Lemma 1. For $\gamma \in \Gamma$ and any site replacement $h : \mathrm{sites}(\gamma) \to \mathcal{S}$, Def. 3 correctly defines composite $\gamma\langle h\rangle$, and, moreover, the following useful relationships are true

$$\begin{aligned} \mathrm{init}(\gamma\langle h\rangle) &= h(\mathrm{init}(\gamma)) \\ \mathrm{term}(\gamma\langle h\rangle) &= h(\mathrm{term}(\gamma)) \\ \mathrm{sites}(\gamma\langle h\rangle) &= h(\mathrm{sites}(\gamma))\,. \end{aligned} \tag{10}$$

Proof.

The proof is by induction on γ.

- By Def. 3, $\lambda\langle h\rangle = \lambda$, hence all the sets in (10) are empty and the equalities hold.

- Let $\gamma = \pi\langle f\rangle$ and $h : \text{sites}(\gamma) \to \mathcal{S}$ be a site replacement. By (3), $f(\text{sites}(\pi)) = \text{sites}(\gamma)$ and $h \circ f : \text{sites}(\pi) \to \mathcal{S}$ is an injective mapping. By Def. 2, $\gamma\langle h\rangle = \pi\langle h \circ f\rangle$ is defined correctly. Next, according to (1–3)

$$\begin{aligned} \text{init}(\pi\langle f\rangle) &= f(\text{init}(\pi)) \\ \text{term}(\pi\langle f\rangle) &= f(\text{term}(\pi)) \\ \text{sites}(\pi\langle f\rangle) &= f(\text{sites}(\pi)) \end{aligned}$$

and

$$\begin{aligned} \text{init}(\pi\langle h \circ f\rangle) &= (h \circ f)(\text{init}(\pi)) = h\big(f(\text{init}(\pi))\big) \\ \text{term}(\pi\langle h \circ f\rangle) &= (h \circ f)(\text{term}(\pi)) = h\big(f(\text{term}(\pi))\big) \\ \text{sites}(\pi\langle h \circ f\rangle) &= (h \circ f)(\text{sites}(\pi)) = h\big(f(\text{sites}(\pi))\big)\,. \end{aligned}$$

From the last equations we obtain (10):

$$\begin{aligned} \text{init}(\pi\langle h \circ f\rangle) &= h(\text{init}(\pi\langle f\rangle)) \\ \text{term}(\pi\langle h \circ f\rangle) &= h(\text{term}(\pi\langle f\rangle)) \\ \text{sites}(\pi\langle h \circ f\rangle) &= h(\text{sites}(\pi\langle f\rangle))\,. \end{aligned} \tag{$*$}$$

- Suppose that for $\alpha \in \Gamma$ the following statement is true: for any site replacement $h' : \text{sites}(\alpha) \to \mathcal{S}$ the composite $\alpha\langle h'\rangle$ has been correctly defined and

$$\begin{aligned} \text{init}(\alpha\langle h'\rangle) &= h'(\text{init}(\alpha)) \\ \text{term}(\alpha\langle h'\rangle) &= h'(\text{term}(\alpha)) \\ \text{sites}(\alpha\langle h'\rangle) &= h'(\text{sites}(\alpha))\,. \end{aligned} \tag{$**$}$$

Let $\gamma = \alpha \lhd \pi\langle f\rangle$ and $h : \text{sites}(\gamma) \to \mathcal{S}$ be a site replacement. It follows from (7) that $\text{sites}(\gamma) = \text{sites}(\alpha) \cup \text{sites}(\pi\langle f\rangle)$. Take $h' = h\big|_{\text{sites}(\alpha)}$. We want to verify (4) for $\alpha\langle h'\rangle$ and $\pi\langle h \circ f\rangle$ in order to check the existence of the composite $\gamma\langle h\rangle = \alpha\langle h'\rangle \lhd \pi\langle h \circ f\rangle$.

$$\begin{aligned} &\text{sites}(\alpha\langle h'\rangle) \cap \text{sites}(\pi\langle h \circ f\rangle) \overset{(*),(**)}{=} \\ &\quad h'(\text{sites}(\alpha)) \cap h(\text{sites}(\pi\langle f\rangle)) = \\ &\quad h\,(\text{sites}(\alpha)) \cap h(\text{sites}(\pi\langle f\rangle)) = \\ &\quad h\,[\text{sites}(\alpha) \cap \text{sites}(\pi\langle f\rangle)] \overset{(4)}{=} \\ &\quad h\,[\text{term}(\alpha) \cap \text{init}(\pi\langle f\rangle)] = \\ &\quad h\,[\text{term}(\alpha)] \cap h[\text{init}(\pi\langle f\rangle)] = \\ &\quad h'(\text{term}(\alpha)) \cap h(\text{init}(\pi\langle f\rangle)) \overset{(*),(**)}{=} \\ &\quad \text{term}(\alpha\langle h'\rangle) \cap \text{init}(\pi\langle h \circ f\rangle)\,. \end{aligned}$$

Finally, we check (10) for $\gamma\langle h\rangle$:

$$\begin{aligned} &\text{init}(\gamma\langle h\rangle) \overset{(5)}{=} \\ &\quad \text{init}(\alpha\langle h'\rangle) \cup [\text{init}(\pi\langle h \circ f\rangle) \setminus \text{term}(\alpha\langle h'\rangle)] \overset{(*),(**)}{=} \\ &\quad h'(\text{init}(\alpha)) \cup [h(\text{init}(\pi\langle f\rangle)) \ \setminus h'(\text{term}(\alpha))] = \\ &\quad h\,(\text{init}(\alpha)) \cup [h(\text{init}(\pi\langle f\rangle)) \ \setminus h\,(\text{term}(\alpha))] = \\ &\quad h\,[\text{init}(\alpha) \cup (\text{init}(\pi\langle f\rangle) \setminus \text{term}(\alpha))] \overset{(5)}{=} \\ &\quad h\,[\text{init}(\alpha \lhd \pi\langle f\rangle)] = \\ &\quad h\,(\text{init}(\gamma))\,. \end{aligned}$$

The proof of the equalities

$$\mathrm{term}(\gamma\langle h\rangle) = h(\mathrm{term}(\gamma))$$
$$\mathrm{sites}(\gamma\langle h\rangle) = h(\mathrm{sites}(\gamma))$$

is similar.

■

Lemma 2. Let γ be a composite, and $h_1 : \mathrm{sites}(\gamma) \to \mathcal{S}$, $h_2 : \mathrm{sites}(\gamma\langle h_1\rangle) \to \mathcal{S}$ be site replacements. Then,

$$(\gamma\langle h_1\rangle)\langle h_2\rangle = \gamma\langle h_2 \circ h_1\rangle.$$

Proof.
The proof is by induction on γ.

- By Def. 3, $(\lambda\langle h_1\rangle)\langle h_2\rangle = \lambda\langle h_2\rangle = \lambda = \lambda\langle h_2 \circ h_1\rangle$.
- Let $\gamma = \pi\langle f\rangle$. Then,

$$(\gamma\langle h_1\rangle)\langle h_2\rangle \overset{\text{D.3}}{=} (\pi\langle h_1 \circ f\rangle)\langle h_2\rangle \overset{\text{D.3}}{=} \pi\langle h_2 \circ (h_1 \circ f)\rangle = \pi\langle (h_2 \circ h_1) \circ f\rangle \overset{\text{D.3}}{=} \pi\langle f\rangle\langle h_2 \circ h_1\rangle.$$

- Suppose that for $\alpha \in \Gamma$ the following statement is true: for all site replacements $g_1 : \mathrm{sites}(\alpha) \to \mathcal{S}$, $g_2 : \mathrm{sites}(\alpha\langle g_1\rangle) \to \mathcal{S}$,

$$(\alpha\langle g_1\rangle)\langle g_2\rangle = \alpha_{.}\langle g_2 \circ g_{1\,.}\rangle \tag{$*$}$$

Let $\gamma = \alpha \lhd \pi\langle f\rangle$, and $h_1 : \mathrm{sites}(\gamma) \to \mathcal{S}$, $h_2 : \mathrm{sites}(\gamma\langle h_1\rangle) \to \mathcal{S}$ be site replacements. Let

$$g_1 = h_1\big|_{\mathrm{sites}(\alpha)}, \quad h = h_2 \circ h_1, \quad g = h_2 \circ g_1 = h\big|_{\mathrm{sites}(\alpha)}.$$

Then,

$$(\gamma\langle h_1\rangle)\langle h_2\rangle \overset{\text{D.3}}{=} (\alpha\langle g_1\rangle \lhd \pi\langle h_1 \circ f\rangle)\langle h_2\rangle \overset{\text{D.3}}{=} (\alpha\langle g_1\rangle)\langle h_2\rangle \lhd \pi\langle h_2 \circ h_1 \circ f\rangle \overset{(*)}{=}$$
$$\alpha\langle h_2 \circ g_1\rangle \lhd \pi\,\langle h_2 \circ h_1 \circ f\,\rangle = \alpha\langle g\rangle \lhd \pi\langle h \circ f\rangle \overset{\text{D.3}}{=} \gamma\langle h\rangle.$$

■

Lemma 3. Let $\gamma' = \gamma\langle h\rangle$. Then, there exists site replacement $h' : \mathrm{sites}(\gamma') \to \mathcal{S}$ such that $\gamma'\langle h'\rangle = \gamma$.

Proof. By Lemma 1, $\mathrm{sites}(\gamma') = h(\mathrm{sites}(\gamma))$. Since h is an injective mapping, let $h' = h^{-1}$, $h' : \mathrm{sites}(\gamma') \to \mathcal{S}$. Then,

$$(\gamma\langle h\rangle)\langle h'\rangle \overset{\text{L.2}}{=} \gamma\langle h' \circ h\rangle = \gamma\langle \mathrm{id}_{\mathrm{sites}(\gamma)}\rangle \overset{\text{D.3}}{=} \gamma.$$

■

Definition 4. Two composites α and β will be called **similar** and denoted as $\alpha \approx \beta$, if there exists site replacement $h : \text{sites}(\beta) \to \mathcal{S}$ such that

$$h\big|_{\text{ext}(\beta)} = \text{id}$$

and

$$\alpha = \beta\langle h \rangle.$$

▶

Thus, two composites are similar, if one of them can be obtained from the other by relabeling its internal sites. The basic relationships between the sites of two similar composites are given next.

Lemma 4. If α and β are two similar composites and h is the corresponding site replacement, then

$$\text{init}(\alpha) = \text{init}(\beta), \quad \text{term}(\alpha) = \text{term}(\beta), \quad \text{int}(\alpha) = h(\text{int}(\beta)).$$

Proof. By Lemma 3,

$$\begin{aligned} \text{init}(\alpha) &= h(\text{init}(\beta)) \\ \text{term}(\alpha) &= h(\text{term}(\beta)) \\ \text{sites}(\alpha) &= h(\text{sites}(\beta)). \end{aligned}$$

Therefore, it follows from (9) that

$$\text{int}(\alpha) = h(\text{int}(\beta)).$$

Again from (9), since $h\big|_{\text{ext}(\beta)} = \text{id}$,

$$\begin{aligned} \text{init}(\alpha) &= \text{init}(\beta) \\ \text{term}(\alpha) &= \text{term}(\beta). \end{aligned}$$

■

How do we construct new composites out of the old ones? The following definition introduces the relevant operation of composition of two composites.

Definition 5. Let α and β be two composites satisfying

$$\text{sites}(\alpha) \cap \text{sites}(\beta) = \text{term}(\alpha) \cap \text{init}(\beta)\,. \tag{11}$$

The **composition** of the above two composites,

$$\alpha \lhd \beta\ ,$$

is defined by induction on β as follows.

- $\alpha \lhd \lambda \overset{\text{def}}{=} \alpha$
- $$\alpha \lhd \pi\langle f \rangle \overset{\text{def}}{=} \begin{cases} \pi\langle f \rangle, & \alpha = \lambda \\ \alpha \lhd \pi\langle f \rangle, & \alpha \neq \lambda \text{ (see Def. 2)} \end{cases}$$

- Assume that $\alpha \lhd \gamma$ has been constructed and that $\beta = \gamma \lhd \pi\langle f\rangle$, then

$$\alpha \lhd \beta \stackrel{\text{def}}{=} (\alpha \lhd \gamma) \lhd \pi\langle f\rangle.$$

▶

Lemma 5. The sets of sites for the composition of two composites α and β are

$$\begin{aligned}
\text{init}(\alpha \lhd \beta) &= \text{init}(\alpha) \cup [\text{init}(\beta) \setminus \text{term}(\alpha)] \\
\text{term}(\alpha \lhd \beta) &= [\text{term}(\alpha) \setminus \text{init}(\beta)] \cup \text{term}(\beta) \\
\text{sites}(\alpha \lhd \beta) &= \text{sites}(\alpha) \cup \text{sites}(\beta) \\
\text{cont}(\alpha \lhd \beta) &= [\text{cont}(\alpha) \cap \text{cont}(\beta)] \cup [\text{cont}(\alpha) \setminus \text{sites}(\beta)] \cup [\text{cont}(\beta) \setminus \text{sites}(\alpha)].
\end{aligned}$$

With respect to definition of the continuation sites (8) one should note that these sites are the only ones that allow the "continuation" of external sites during the composition.

Lemma 6. Composition is associative, i.e. if α, β, γ are composites such that compositions $\alpha \lhd \beta$ and $(\alpha \lhd \beta) \lhd \gamma$ exist, then the compositions $\beta \lhd \gamma$ and $\alpha \lhd (\beta \lhd \gamma)$ also exist and

$$(\alpha \lhd \beta) \lhd \gamma = \alpha \lhd (\beta \lhd \gamma)\,.$$

2.2 Semantic relations, istructs, and parallel composition

As it often happens in science, the accumulated (mainly experimentally) knowledge in a particular domain strongly suggests the "indistinguishability" of some objects or their parts in the following sense: if the corresponding two parts in the object representation are interchanged, no visible/detectable differences in the "behaviour" of the reference physical objects are observed. It is for this reason that we, as always, need the following concept.

Definition 6. Let α, β be two composites such that

$$\text{init}(\alpha) = \text{init}(\beta), \qquad \text{term}(\alpha) = \text{term}(\beta).$$

The expression

$$\alpha \equiv \beta$$

is called **semantic identity** and signifies the indistinguishability of the corresponding two parts in an object representation (Fig 6). ▶

Definition 7. Let $\mathcal{I}$ be a specified set of semantic identities.[11] This set induces naturally the semantic equivalence relation, or simply **semantic relation**, denoted $\sim$, on the set of composites Γ as follows.

[11] Note that, usually, $\mathcal{I}$ is a relatively small set.

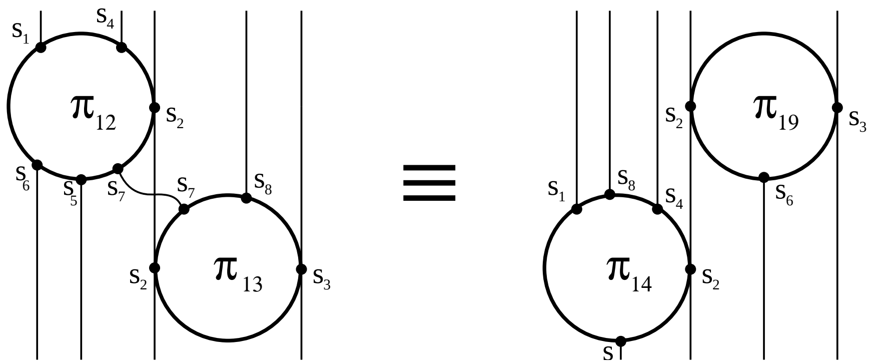

Figure 6: Example of a semantic identity.

1. If $\alpha \equiv \beta$ is a semantic identity (from $\mathcal{I}$), then $\alpha \sim \beta$.

2. If $\alpha \sim \beta$ and

$$f : \text{sites}(\alpha) \to \mathcal{S}$$
$$g : \text{sites}(\beta) \to \mathcal{S}$$

   are **externally consistent** site replacements, i.e.

$$f\big|_{\text{ext}(\alpha)} = g\big|_{\text{ext}(\beta)} ,$$

   then

$$\alpha\langle f\rangle \ \sim \ \beta\langle g\rangle.$$

3. If $\alpha \sim \beta$, $\gamma \sim \delta$ and compositions $\alpha \lhd \gamma$, $\beta \lhd \delta$ exist, then

$$\alpha \lhd \gamma \ \sim \ \beta \lhd \delta.$$

4. Finally, the binary relation $\sim$ is defined as the minimal equivalence relation satisfying the above two conditions, i.e. it is the intersection of all the equivalence relations satisfying the above conditions.

▶

Note that if $\mathcal{I} = \varnothing$, then the semantic relation becomes the similarity relation (see Def. 4).

Lemma 7. If two composites are semantically equivalent, $\alpha \sim \beta$, then their sets of initial and terminal sites are identical.

Proof. Let

$$B = \{(\alpha, \beta) \mid \text{init}(\alpha) = \text{init}(\beta), \ \ \text{term}(\alpha) = \text{term}(\beta)\}.$$

Note that B is an equivalence relation on Γ. Let us check that B satisfies 1–3 from Def. 7.

1. If $\alpha \equiv \beta$, then, by Def. 6, $(\alpha, \beta) \in B$.

2. If $(\alpha, \beta) \in B$ and f, g are the corresponding externally consistent site replacements, then

$$\operatorname{init}(\alpha\langle f\rangle) \overset{\text{L.1}}{=} f(\operatorname{init}(\alpha)) \overset{(9)}{=} f\big|_{\operatorname{ext}(\alpha)}(\operatorname{init}(\alpha)) = g\big|_{\operatorname{ext}(\beta)}(\operatorname{init}(\alpha)) \overset{(\alpha,\beta)\in B}{=}$$
$$g\big|_{\operatorname{ext}(\beta)}(\operatorname{init}(\beta)) \overset{(9)}{=} g(\operatorname{init}(\beta)) \overset{\text{L.1}}{=} \operatorname{init}(\beta\langle g\rangle)\,,$$

and, similarly,

$$\operatorname{term}(\alpha\langle f\rangle) = \operatorname{term}(\beta\langle g\rangle).$$

Hence, $(\alpha\langle f\rangle, \beta\langle g\rangle) \in B$.

3. If $(\alpha, \beta),\ (\gamma, \delta) \in B$, and the compositions $\alpha \lhd \gamma$, $\beta \lhd \delta$ exist, then

$$\begin{aligned} \operatorname{init}(\alpha \lhd \gamma) \overset{\text{L.5}}{=}\ & \operatorname{init}(\alpha) \cup [\operatorname{init}(\gamma) \setminus \operatorname{term}(\alpha)] = \\ & \operatorname{init}(\beta) \cup [\operatorname{init}(\delta) \setminus \operatorname{term}(\beta)] \overset{\text{L.5}}{=} \operatorname{init}(\beta \lhd \delta). \end{aligned}$$

and, similarly,

$$\operatorname{term}(\alpha \lhd \gamma) = \operatorname{init}(\beta \lhd \delta).$$

Hence, $(\alpha \lhd \gamma, \beta \lhd \delta) \in B$.

From condition 4 of Def. 7, we have $\sim \subseteq B$. ■

Lemma 8. If $\alpha \approx \beta$, then $\alpha \sim \beta$.

One of the simplest semantic identities specifies the indistinguishability between some of the sites in a primitive.

Definition 8. For primitive $\pi\langle f\rangle$ and site replacement $h : \operatorname{sites}(\pi\langle f\rangle) \to \mathcal{S}$ satisfying

$$h[\operatorname{init}(\pi\langle f\rangle)] = \operatorname{init}(\pi\langle f\rangle)$$
$$h[\operatorname{term}(\pi\langle f\rangle)] = \operatorname{term}(\pi\langle f\rangle)\,,$$

the semantic identity

$$\pi\langle f\rangle \ \equiv\ \pi\langle h \circ f\rangle$$

will be called the **site equivalence identity**. We will denote by $\operatorname{Eqsite}(\Pi)$ the set of all possible such identities. ▶

Judging by our present knowledge in physics, i.e. by the indistinguishability between any two elementary particles of the same type, the set $\operatorname{Eqsite}(\Pi)$ or its generalization might be of particular importance in a large number of applications.

It is not difficult to anticipate now that, as always is the case, the set of equivalent composites should be considered as representing the same physical object.

Definition 9. Let Π, $\mathcal{I}$ be specified sets of primtypes and semantic identities. The quotient set

$$\Theta = \Gamma/_{\sim} = \{\, [\gamma] \,|\, \gamma \in \Gamma \,\}$$

will be called the set of instance structs, or simply **istructs** (for $(\Pi, \mathcal{I})$). Istruct $[\lambda]$ will be called the **empty istruct** and denoted $\boldsymbol{\lambda}$. For each istruct $[\gamma]$, also denoted, $\boldsymbol{\gamma}$ the three sets of sites are defined as follows

$$\mathrm{init}(\boldsymbol{\gamma}) = \mathrm{init}(\gamma), \quad \mathrm{term}(\boldsymbol{\gamma}) = \mathrm{term}(\gamma), \quad \mathrm{ext}(\boldsymbol{\gamma}) = \mathrm{ext}(\gamma)\ .$$

▶

Note that the correctness of the last definition follows from Lemma 7.

The concept of quotient set plays an important role in the proposed representation model: the concept of istruct should be considered as the first step in the process of connecting the "abstract" representation by means of composites with the "actual" objects.[12] In view of this, it might be useful to remember how to work with quotient sets. Some of the most known examples of equivalence relations are rational numbers ($1/2 \sim 2/4 \sim \ldots$) and systems of (linear) equations (systems are equivalent, iff they have the same solutions). So, in order to work with the quotient set, one should be able, first, to find (algorithmically) a "canonical" element of the equivalence class (which represents the class), and, second, learn how to operate with the original set (add, multiply, and, in our case, attach or replace sites). Note that, although the *general* problem of finding a canonical composite is undecidable, in each application of the proposed model one should make sure that there is an *efficient* algorithm for finding the canonical composite for any istruct.

Definition 10. For an istruct $\boldsymbol{\gamma}$ and an injective mapping $\mathbf{h} : \mathrm{ext}(\boldsymbol{\gamma}) \to \mathcal{S}$, called **istruct site replacement**, the istruct $\boldsymbol{\gamma}\langle\mathbf{h}\rangle$ is defined as

$$\boldsymbol{\gamma}\langle\mathbf{h}\rangle = [\gamma\langle h\rangle]\ ,$$

where $\gamma \in \boldsymbol{\gamma}$ and $h : \mathrm{sites}(\gamma) \to \mathcal{S}$ is a site replacement satisfying

$$h\big|_{\mathrm{ext}(\gamma)} = \mathbf{h}.$$

▶

Note that although a composite, typically, has internal sites, we do not introduce the concept of internal sites for an istruct, since an istruct is "independent" of the labels of the internal sites in its canonical composite. Also, as was the case with composites, in general, we cannot arbitrarily modify the labels of the external sites of an istruct, since the modification may change the manner of attachment of this istruct to other istructs.

Lemma 9. For an istruct $\boldsymbol{\gamma}$ and an injective mapping $\mathbf{h} : \mathrm{ext}(\boldsymbol{\gamma}) \to \mathcal{S}$, the istruct $\boldsymbol{\gamma}\langle\mathbf{h}\rangle$ is correctly defined in Def. 10, i.e. it does not depend on the choice of γ and h.

Proof. Let $\gamma_1, \gamma_2 \in \boldsymbol{\gamma}$, and $h_1 : \mathrm{sites}(\gamma_1) \to \mathcal{S}$, $h_2 : \mathrm{sites}(\gamma_2) \to \mathcal{S}$ be site replacements satisfying

$$h_1\big|_{\mathrm{ext}(\gamma_1)} = h_2\big|_{\mathrm{ext}(\gamma_2)} = \mathbf{h},$$

i.e. h_1 and h_2 are externally consistent. By Def. 9, $\gamma_1 \sim \gamma_2$. Therefore, according to Def. 7.2, $\gamma_1\langle h_1\rangle \sim \gamma_2\langle h_2\rangle$. ■

[12] For the next step, see the concept of struct introduced in Def. 21.

Lemma 10. Let γ be a composite, and $\mathbf{h}_1 : \mathrm{ext}(\boldsymbol{\gamma}) \to \mathcal{S}$, $\mathbf{h}_2 : \mathrm{ext}(\boldsymbol{\gamma}\langle\mathbf{h}_1\rangle) \to \mathcal{S}$ be site replacements. Then,

$$(\boldsymbol{\gamma}\langle\mathbf{h}_1\rangle)\langle\mathbf{h}_2\rangle = \boldsymbol{\gamma}\langle\mathbf{h}_2 \circ \mathbf{h}_1\rangle.$$

Proof. Take any $\gamma \in \boldsymbol{\gamma}$, $h_1 : \mathrm{sites}(\gamma) \to \mathcal{S}$, $h_2 : \mathrm{sites}(\gamma\langle h_1\rangle) \to \mathcal{S}$ satisfying

$$h_1\big|_{\mathrm{ext}(\boldsymbol{\gamma})} = \mathbf{h}_1, \qquad h_2\big|_{\mathrm{ext}\,(\boldsymbol{\gamma}\langle\mathbf{h}_1\rangle)} = \mathbf{h}_2.$$

Then,

$$(\boldsymbol{\gamma}\langle\mathbf{h}_1\rangle)\langle\mathbf{h}_2\rangle \overset{\text{D.10}}{=} [(\gamma\langle h_1\rangle)\langle h_2\rangle] \overset{\text{L.2}}{=} [\gamma\langle h_2 \circ h_1\rangle] \overset{\text{D.10}}{=} \boldsymbol{\gamma}\langle\mathbf{h}_2 \circ \mathbf{h}_1\rangle.$$

■

Lemma 11. Let $\boldsymbol{\gamma}' = \boldsymbol{\gamma}\langle\mathbf{h}\rangle$. Then, there exists site replacement $\mathbf{h}' : \mathrm{ext}(\boldsymbol{\gamma}') \to \mathcal{S}$ such that $\boldsymbol{\gamma}'\langle\mathbf{h}'\rangle = \boldsymbol{\gamma}$.

Proof. The proof of this lemma is similar to that of Lemma 3. ■

Lemma 12. If $\boldsymbol{\alpha}\langle\mathbf{h}\rangle = \boldsymbol{\beta}\langle\mathbf{h}\rangle$, then $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

Lemma 13. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a finite subset of the set of sites. Let $\boldsymbol{\gamma}$ be an istruct, and $\{\boldsymbol{\alpha}_i\}_{i=1}^{\infty}$ be a sequence of istructs, $\{\mathbf{h}_i\}_{i=1}^{\infty}$ be a sequence of site replacements such that for all $i \in \mathbb{Z}_+$,

- $\mathrm{ext}(\boldsymbol{\alpha}_i) \subseteq \mathcal{S}'$
- $\boldsymbol{\gamma} = \boldsymbol{\alpha}_i\langle\mathbf{h}_i\rangle$.

Then there exist indices $i \neq j$ such that $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_j$.

Proof. For each $i \in \mathbb{Z}_+$, the domain of $\mathbf{h}_i$ is a subset of $\mathcal{S}'$ and the image of $\mathbf{h}_i$ is equal to $\mathrm{ext}(\boldsymbol{\gamma})$. Both, $\mathcal{S}'$ and $\mathrm{ext}(\boldsymbol{\gamma})$ are finite sets. Therefore, there exist indices $i \neq j$, such that $\mathbf{h}_i = \mathbf{h}_j$, and, by Lemma 12, we have $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_j$. ■

As we did for the composites, we now introduce the concept of composition for istructs.

Definition 11. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be two istructs satisfying

$$\mathrm{ext}(\boldsymbol{\alpha}) \cap \mathrm{ext}(\boldsymbol{\beta}) = \mathrm{term}(\boldsymbol{\alpha}) \cap \mathrm{init}(\boldsymbol{\beta})\,.$$

The **composition of** the above two **istructs**, $\boldsymbol{\alpha} \lhd \boldsymbol{\beta}$, is defined as

$$\boldsymbol{\alpha} \lhd \boldsymbol{\beta} = [\alpha \lhd \beta]\,,$$

where

$$\alpha \in \boldsymbol{\alpha}, \quad \beta \in \boldsymbol{\beta}, \quad \text{and} \quad \alpha \lhd \beta \ \text{exists.}$$

▶

Lemma 14. The composition of the above istructs $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is correctly defined, i.e. it does not depend on the choice of α and β, and exists.

Proof. To prove the existence, we need to find $\gamma \in \boldsymbol{\alpha}$ and $\delta \in \boldsymbol{\beta}$, such that $\gamma \lhd \delta$ exists. Take any $\alpha \in \boldsymbol{\alpha}$, $\beta \in \boldsymbol{\beta}$. There exist site replacements

$$f : \mathrm{sites}(\alpha) \to \mathcal{S}, \quad g : \mathrm{sites}(\beta) \to \mathcal{S}$$

satisfying

$$f\big|_{\mathrm{ext}(\alpha)} = \mathrm{id}, \quad g\big|_{\mathrm{ext}(\beta)} = \mathrm{id}$$

and

$$\begin{aligned} \mathrm{int}(\alpha\langle f\rangle) \cap \mathrm{sites}(\beta\langle g\rangle) = \varnothing \\ \mathrm{int}(\beta\langle g\rangle) \cap \mathrm{sites}(\alpha\langle f\rangle) = \varnothing. \end{aligned} \tag{$*$}$$

Note that by Def. 7,

$$\alpha\langle f\rangle \sim \alpha \quad \text{and} \quad \beta\langle g\rangle \sim \beta, \quad \text{or,}$$

$$\alpha\langle f\rangle \in \boldsymbol{\alpha} \quad \text{and} \quad \beta\langle g\rangle \in \boldsymbol{\beta}. \tag{$**$}$$

Therefore,

$$\begin{aligned} &\mathrm{sites}(\alpha\langle f\rangle) \cap \mathrm{sites}(\beta\langle g\rangle) \overset{(9)}{=} \\ &[\mathrm{ext}(\alpha\langle f\rangle) \cup \mathrm{int}(\alpha\langle f\rangle)] \cap [\mathrm{ext}(\beta\langle g\rangle) \cup \mathrm{int}(\beta\langle g\rangle)] \overset{(*)}{=} \\ &\mathrm{ext}(\alpha\langle f\rangle) \cap \mathrm{ext}(\beta\langle g\rangle) \overset{(**)}{=} \mathrm{ext}(\boldsymbol{\alpha}) \cap \mathrm{ext}(\boldsymbol{\beta}) \overset{\text{Def.11}}{=} \\ &\mathrm{term}(\boldsymbol{\alpha}) \cap \mathrm{init}(\boldsymbol{\beta}) \overset{(**)}{=} \mathrm{term}(\alpha\langle f\rangle) \cap \mathrm{init}(\beta\langle g\rangle), \end{aligned}$$

and it follows from (4) that the composition $\alpha\langle f\rangle \lhd \beta\langle g\rangle$ exists.

If $\alpha \sim \alpha'$, $\beta \sim \beta'$, then by Def. 7

$$\alpha \lhd \beta \ \sim \ \alpha' \lhd \beta',$$

and so $\boldsymbol{\alpha} \lhd \boldsymbol{\beta}$ does not depend on the choice of α and β, i.e. the correctness of Def. 11 is proved. ■

Lemma 15. Composition of istructs is associative.

The next two definitions single out a certain subclass of compositions that can be thought of as representing "independent", or "parallel", attachment of composites.

Definition 12. For $\alpha, \beta \in \Gamma$, for which

$$\mathrm{sites}(\alpha) \cap \mathrm{sites}(\beta) = \mathrm{cont}(\alpha) \cap \mathrm{cont}(\beta)\,, \tag{12}$$

the composite $\alpha \lhd \beta$ will be denoted as

$$\alpha \,\|\, \beta$$

and called the **parallel composition of α and β.** ▶

Lemma 16. The above definition is correct, i.e. from (12) it follows that parallel compositions $\alpha \,\|\, \beta$ and $\beta \,\|\, \alpha$ exist. Moreover,

$$\begin{aligned} \mathrm{init}(\alpha \,\|\, \beta) &= \mathrm{init}(\alpha) \cup \mathrm{init}(\beta) \\ \mathrm{term}(\alpha \,\|\, \beta) &= \mathrm{term}(\alpha) \cup \mathrm{term}(\beta) \\ \mathrm{cont}(\alpha \,\|\, \beta) &= \mathrm{cont}(\alpha) \cup \mathrm{cont}(\beta). \end{aligned} \tag{13}$$

Definition 13. If

$$\alpha = \pi_1\langle f_1\rangle \parallel \pi_2\langle f_2\rangle \ , \quad \beta = \pi_2\langle f_2\rangle \parallel \pi_1\langle f_1\rangle \ ,$$

then the semantic identity

$$\alpha \equiv \beta$$

is called the **commutativity identity**. We will denote by Comm(Π) the set of all *possible* such identities (See Fig. 7). ►

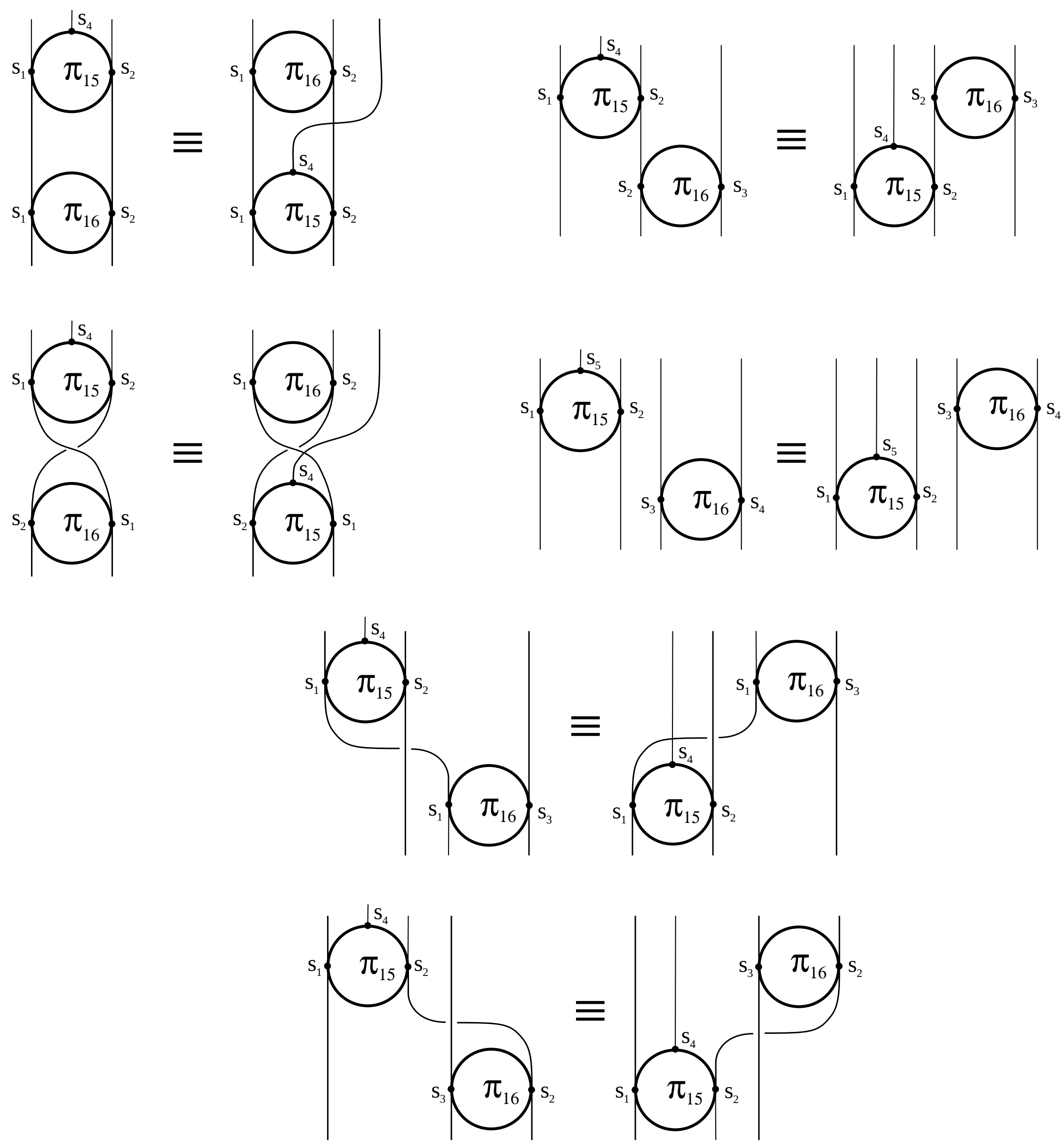

Figure 7: Example of Comm(Π) for $\Pi = \{\pi_{15}, \pi_{16}\}$, each of which has two continuation sites.

Lemma 17. Let $\mathcal{I} \supseteq \mathrm{Comm}(\Pi)$. If for composites α, β the parallel composition $\alpha \,\|\, \beta$ exists, then

$$\alpha \,\|\, \beta \;\sim\; \beta \,\|\, \alpha.$$

Lemma 18. Let $\mathcal{I} = \mathrm{Comm}(\Pi)$ and B be the set of *all* bijections $b : [1, n] \to [1, n]$. If $\gamma = \pi_1\langle f_1\rangle \,\|\, \ldots \,\|\, \pi_n\langle f_n\rangle$, then $\mathrm{int}(\gamma) = \varnothing$ and

$$\boldsymbol{\gamma} = \left\{\; \pi_{b(1)}\langle f_{b(1)}\rangle \,\|\, \ldots \,\|\, \pi_{b(n)}\langle f_{b(n)}\rangle \;\middle|\; b \in B \;\right\}.$$

2.3 Itransformations

In this section, we make the next big step by introducing the next central concept, the concept of istruct transformation, i.e. the concept of istruct "operation".

Definition 14. Let $(\Pi, \mathcal{I})$ be given and let Θ be the corresponding set of istructs. A pair $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ of istructs such that there exists istruct $\boldsymbol{\delta}$ satisfying

$$\boldsymbol{\beta} = \boldsymbol{\alpha} \lhd \boldsymbol{\delta}\,,$$

will be called an instance transformation, or simply **itransformation**, $\boldsymbol{\alpha}$ will be called the **context of itransformation** $\boldsymbol{\tau}$, and

$$\mathrm{ext}(\boldsymbol{\tau}) \stackrel{\mathrm{def}}{=} \mathrm{ext}(\boldsymbol{\alpha}) \cup \mathrm{ext}(\boldsymbol{\beta}).$$

If $\boldsymbol{\alpha} = [\lambda]$, the itransformation will be called **context free**. The set of all itransformations for $(\Pi, \mathcal{I})$ will be denoted by $\mathcal{T}$ (See Fig. 8). ▶

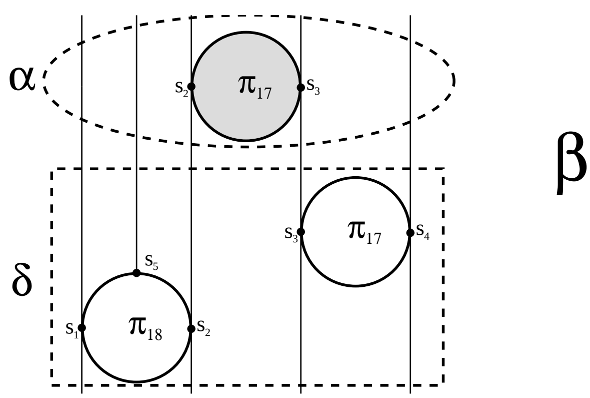

Figure 8: Pictorial represenatation of itransformation $\boldsymbol{\tau} = (\,[\,\pi_{17}\langle f_{17}\rangle\,],\,[\,\pi_{17}\langle f_{17}\rangle \lhd \pi_{17}\langle g_{17}\rangle \lhd \pi_{18}\langle f_{18}\rangle\,]\,)$. The shaded composite $\pi_{17}\langle f_{17}\rangle$ is an element from the context $[\pi_{17}\langle f_{17}\rangle]$.

As was the case with composites and istructs, it is useful to introduce the concept of itransformation site replacement.

Definition 15. For an itransformation $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ and an injective mapping $\mathbf{h} : \mathrm{ext}(\boldsymbol{\tau}) \to \mathcal{S}$, called the **itransformation site replacement**, the itransformation $\boldsymbol{\tau}\langle\mathbf{h}\rangle$ is defined as

$$\boldsymbol{\tau}\langle\mathbf{h}\rangle \stackrel{\text{def}}{=} (\boldsymbol{\alpha}\langle\mathbf{h}\big|_{\mathrm{ext}(\boldsymbol{\alpha})}\rangle, \boldsymbol{\beta}\langle\mathbf{h}\big|_{\mathrm{ext}(\boldsymbol{\beta})}\rangle).$$

▶

Definition 16. (See Fig. 9.) For an istruct $\boldsymbol{\gamma}$ and itransformation $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfying

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}_{\text{front}} \lhd \boldsymbol{\alpha},$$

the **$\boldsymbol{\tau}$-itransformation of istruct** $\boldsymbol{\gamma}$, denoted $\boldsymbol{\gamma} \lhd \boldsymbol{\tau}$, is defined as the istruct

$$\boldsymbol{\gamma}_{\text{front}} \lhd \boldsymbol{\beta}.$$

▶

Remark 1. Note that usually $\boldsymbol{\tau}$-itransformation of $\boldsymbol{\gamma}$ is not affected by the site replacements of $\boldsymbol{\tau}$ that modifies only the initial sites of the context $\boldsymbol{\alpha}$.

Lemma 19. The above definition is correct, i.e. $\boldsymbol{\gamma} \lhd \boldsymbol{\tau}$ does not depend on the choice of $\boldsymbol{\gamma}_{\text{front}}$.

Proof. Let $\boldsymbol{\gamma} = \boldsymbol{\gamma}_1 \lhd \boldsymbol{\alpha} = \boldsymbol{\gamma}_2 \lhd \boldsymbol{\alpha}$. By Def. 14, there exists istruct $\boldsymbol{\varepsilon}$ such that $\boldsymbol{\beta} = \boldsymbol{\alpha} \lhd \boldsymbol{\varepsilon}$. Therefore,

$$\boldsymbol{\gamma}_1 \lhd \boldsymbol{\beta} = \boldsymbol{\gamma}_1 \lhd (\boldsymbol{\alpha} \lhd \boldsymbol{\varepsilon}) \stackrel{\text{L.15}}{=} (\boldsymbol{\gamma}_1 \lhd \boldsymbol{\alpha}) \lhd \boldsymbol{\varepsilon} = (\boldsymbol{\gamma}_2 \lhd \boldsymbol{\alpha}) \lhd \boldsymbol{\varepsilon} \stackrel{\text{L.15}}{=} \boldsymbol{\gamma}_2 \lhd (\boldsymbol{\alpha} \lhd \boldsymbol{\varepsilon}) = \boldsymbol{\gamma}_2 \lhd \boldsymbol{\beta}.$$

■

Lemma 20. Let $\boldsymbol{\gamma}, \boldsymbol{\gamma}'$ be istructs and $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an itransformation. If $\boldsymbol{\gamma}' = \boldsymbol{\gamma} \lhd \boldsymbol{\tau}$, then there exists istruct $\boldsymbol{\delta}$ such that $\boldsymbol{\gamma}' = \boldsymbol{\gamma} \lhd \boldsymbol{\delta}$.

Proof. By Def. 14, there exists $\boldsymbol{\delta}$ such that $\boldsymbol{\beta} = \boldsymbol{\alpha} \lhd \boldsymbol{\delta}$. By Def. 16, there exists $\boldsymbol{\varepsilon}$ such that $\boldsymbol{\gamma} = \boldsymbol{\varepsilon} \lhd \boldsymbol{\alpha}$, and

$$\boldsymbol{\gamma}' = \boldsymbol{\varepsilon} \lhd \boldsymbol{\beta} = \boldsymbol{\varepsilon} \lhd (\boldsymbol{\alpha} \lhd \boldsymbol{\delta}) \stackrel{\text{L. 15}}{=} (\boldsymbol{\varepsilon} \lhd \boldsymbol{\alpha}) \lhd \boldsymbol{\delta} = \boldsymbol{\gamma} \lhd \boldsymbol{\delta}.$$

■

Lemma 21. Let $\boldsymbol{\gamma}, \boldsymbol{\gamma}', \boldsymbol{\varepsilon}, \boldsymbol{\rho}$ be istructs and $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ be an itransformation. If $\boldsymbol{\gamma}' = \boldsymbol{\gamma} \lhd \boldsymbol{\tau}$ and $\boldsymbol{\rho} = \boldsymbol{\varepsilon} \lhd \boldsymbol{\gamma}$, then

$$\boldsymbol{\rho} \lhd \boldsymbol{\tau} = \boldsymbol{\varepsilon} \lhd \boldsymbol{\gamma}'.$$

Proof. Since $\boldsymbol{\gamma}' = \boldsymbol{\gamma} \lhd \boldsymbol{\tau}$, there exists istruct $\boldsymbol{\gamma}_{\text{front}}$ such that $\boldsymbol{\gamma} = \boldsymbol{\gamma}_{\text{front}} \lhd \boldsymbol{\alpha}$ (by Def. 16). Then,

$$\boldsymbol{\rho} = \boldsymbol{\varepsilon} \lhd \boldsymbol{\gamma} = \boldsymbol{\varepsilon} \lhd (\boldsymbol{\gamma}_{\text{front}} \lhd \boldsymbol{\alpha}) \stackrel{\text{L. 15}}{=} (\boldsymbol{\varepsilon} \lhd \boldsymbol{\gamma}_{\text{front}}) \lhd \boldsymbol{\alpha},$$

therefore,

$$\boldsymbol{\rho} \lhd \boldsymbol{\tau} = (\boldsymbol{\varepsilon} \lhd \boldsymbol{\gamma}_{\text{front}}) \lhd \boldsymbol{\beta} \stackrel{\text{L. 15}}{=} \boldsymbol{\varepsilon} \lhd (\boldsymbol{\gamma}_{\text{front}} \lhd \boldsymbol{\beta}) = \boldsymbol{\varepsilon} \lhd \boldsymbol{\gamma}'.$$

■

According to the above definition, the action of an itransformation $\boldsymbol{\tau}$ can be viewed as an "attachment" of $\boldsymbol{\tau}$ to some istruct which contains the corresponding context. Thus, one can think of an istruct as of a representation of the object's constructive history (via the history of applied itransformations). To put it more precisely, an istruct represents all possible indistinguishable construction pathways for a given object. An itransformation context is a part of the istruct, it represents a part of the object's constructive history. The itransformation does not "delete" any information from the istruct, and therefore it can be thought of as an *evolutionary* transformation.

One should also emphasize some basic distinctions (of transformations) from Chomsky production rules.

- Although the context of a Chomsky production rule (regardless of whether it is a context-free, context-sensitive or unrestricted production) is a part of the string, since the string does not capture its constructive history, the production cannot refer to the corresponding part of the history.
- When a production is applied, the non-terminal symbol, which served as the production context, is erased, and cannot serve as a context for further productions. Therefore, in Chomsky's model, productions erase the string's evolutionary history.

It is interesting to note that in [23] Chomsky speaks of certain grammatical rules which can be applied to sentences only if we "... know not only the final shape of these sentences but also their 'history of derivation'." He does not offer, however, any *formal definition* of such rules.

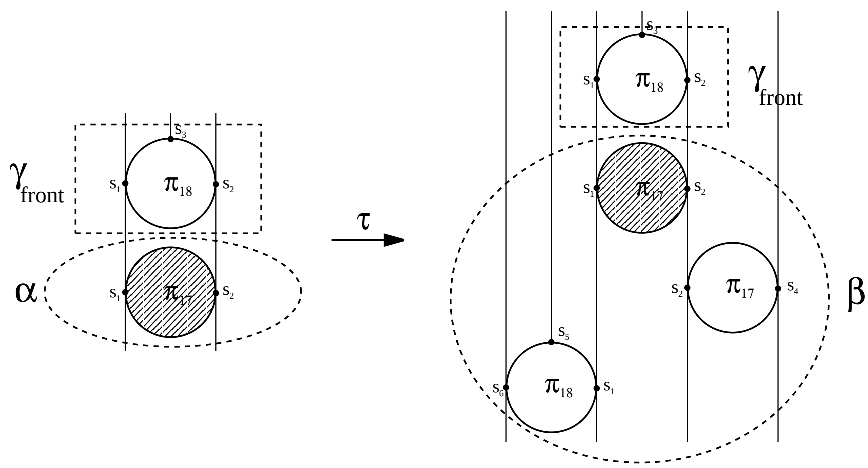

Figure 9: Example of $\boldsymbol{\tau}$-itransformation (from Fig. 8) of istruct $\boldsymbol{\gamma} = [\pi_{18}\langle g_{18}\rangle \triangleleft \pi_{17}\langle f_{17}\rangle]$. The shaded parts in the representative composites correspond to the context of $\boldsymbol{\tau}$.

Definition 17. If $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \Theta,\ \boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{T}$, and there exists a site replacement $\mathbf{h}$ such that

$$\boldsymbol{\gamma}' = \boldsymbol{\gamma} \triangleleft \boldsymbol{\tau}\langle \mathbf{h}\rangle,$$

then we will signify it

$$\boldsymbol{\gamma} \xrightarrow{\boldsymbol{\tau}} \boldsymbol{\gamma}' \,.$$

►

As in the classical case (for which we recommend [24]), to formulate the axiom of induction we need an auxiliary definition.

Definition 18. The **set of immediate ancestors of an istruct $\boldsymbol{\gamma}$** is defined as

$$\mathcal{AI}(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \{\, \boldsymbol{\alpha} \in \Theta \mid (\boldsymbol{\alpha} \neq \boldsymbol{\gamma}) \text{ and } (\exists \boldsymbol{\tau} \in \mathcal{T} \quad \boldsymbol{\alpha} \xrightarrow{\boldsymbol{\tau}} \boldsymbol{\gamma}) \,\} \,.$$

▶

Definition 19. (Induction axiom for istructs.) If Θ' is a subset of Θ satisfying the following conditions

- $\boldsymbol{\lambda} \in \Theta'$
- $\forall \boldsymbol{\gamma} \in \Theta \quad [\, \mathcal{AI}(\boldsymbol{\gamma}) \subseteq \Theta' \implies \boldsymbol{\gamma} \in \Theta' \,]$,

then $\Theta' = \Theta$. ▶

This axiom is necessary whenever a proof or definition by induction on istructs is required.

We are ready now to introduce our underlying, or basic, mathematical structure.

Definition 20. Let Π be a finite set of primtypes, $\mathcal{I}$ be a specified set of semantic identities, and Θ, $\mathcal{T}$ be the corresponding sets of istructs and itransformations. If the induction axiom for istructs holds, we will call $(\Pi, \mathcal{I})$ an **inductive structure**. ▶

2.4 The string inductive structure

In this and the following examples, function $f : [1, n] \to \mathbb{Z}_+$ will be specified by the list of its values $f(1), \ldots, f(n)$. Also, for simplicity, we will use the same notation if $f : \{a_1, a_2, \ldots, a_n\} \to \mathbb{Z}_+$, where the domain of f is a finite subset of set A of abstract sites (see Def. 1).

Example 1. Let $\Sigma = \{a, b, \ldots, z\}$ be a finite alphabet. Consider a set of primtypes $\Pi = \{\pi_a, \pi_b, \ldots, \pi_z\}$ indexed by the elements of Σ. Let the set A of a-sites be $\{a_1, a_2, a_3\}$, and for each $\pi \in \Pi$,

$$\begin{aligned} \text{init}(\pi) &= \{a_1\} \\ \text{term}(\pi) &= \{a_2, a_3\}. \end{aligned}$$

The set S of sites is $\mathbb{N}$. Then, each composite corresponds to a string or to a finite set of strings [13] over Σ. E.g., both composites $\pi_a\langle 1,2,3\rangle \lhd \pi_b\langle 3,4,5\rangle$ and $\pi_b\langle 1,3,5\rangle \lhd \pi_a\langle 3,2,4\rangle$ correspond to the same string ab. However, each composite represents different "construction path" for this string: the first composite corresponds to the concatenation of letter b after a, and the second one, to the concatenation of a in front of b.

To model "strings", we would like to model the situation where these construction paths are equivalent. To this end, let us, for each $\alpha, \beta \in \Sigma$, denote by $\text{id}_{\alpha\beta}$ the following semantic identity:

$$\pi_\alpha\langle 1,2,3\rangle \lhd \pi_\beta\langle 3,4,5\rangle \equiv \pi_\beta\langle 1,3,5\rangle \lhd \pi_\alpha\langle 3,2,4\rangle.$$

[13] This is the case when the composite is "decoupled" (see Fig. 4).

Let
$$\mathcal{I} = \text{Comm}(\Pi) \cup \{\text{id}_{\alpha\beta} \mid \alpha, \beta \in \Sigma\}.$$

Then, each istruct represents the set of *all* construction paths for a given string (or a set of strings). E.g., istruct
$$[\pi_b\langle 1,3,5\rangle \lhd \pi_a\langle 3,2,4\rangle]$$
represents *all* construction paths for the string ab, and, therefore, could be thought of as the string ab itself. We will denote by $\text{str}(\boldsymbol{\gamma})$ the string (or the set of strings) corresponding to istruct $\boldsymbol{\gamma}$. Below, for simplicity, we will restrict ourselves to istructs that correspond to one string only.

Each itransformation $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ corresponds to a *set of context-dependent insertions of substrings:*

- $\text{str}(\boldsymbol{\alpha})$ is a subsequence of $\text{str}(\boldsymbol{\beta})$
- if itransformation $\boldsymbol{\tau}$ is applicable to istruct $\boldsymbol{\gamma}$, then $\text{str}(\boldsymbol{\alpha})$ is a substring of $\text{str}(\boldsymbol{\gamma})$, i.e. $\text{str}(\boldsymbol{\gamma}) = u\,\text{str}(\boldsymbol{\alpha})v$ for some $u, v \in \Sigma^*$
- $\text{str}(\boldsymbol{\gamma} \lhd \boldsymbol{\tau}) = u\,\text{str}(\boldsymbol{\beta})v$.

We will refer to the above inductive structure as the *string inductive structure* (SIS). ♢

3 Transformation systems and classes

In this section we will assume that some inductive structure $(\Pi, \mathcal{I})$ is fixed. Moreover, note that each example builds on the previous ones.

3.1 Structs, transformations, transformation sets

The next definition, as was mentioned above, completes the process of connecting the "abstract" representation (by means of composites) with the "actual" objects, or structs.

Definition 21. For an istruct $\boldsymbol{\gamma} \in \Theta$, the set
$$\bar{\boldsymbol{\gamma}} \overset{\text{def}}{=} \{\boldsymbol{\gamma}\langle\mathbf{h}\rangle \mid \mathbf{h} \text{ is a site replacement}\}$$
will be called the **struct** corresponding to $\boldsymbol{\gamma}$.
For an itransformation $\boldsymbol{\tau} \in \mathcal{T}$, the set
$$\bar{\boldsymbol{\tau}} \overset{\text{def}}{=} \{\boldsymbol{\tau}\langle\mathbf{h}\rangle \mid \mathbf{h} \text{ is a site replacement}\}$$
will be called the **transformation** corresponding to $\boldsymbol{\tau}$.
The sets of structs and transformations, in the inductive structure $(\Pi, \mathcal{I})$, will be denoted as $\bar{\Theta} \overset{\text{def}}{=} \{\bar{\boldsymbol{\gamma}} \mid \boldsymbol{\gamma} \in \Theta\}$ and $\bar{\mathcal{T}} \overset{\text{def}}{=} \{\bar{\boldsymbol{\tau}} \mid \boldsymbol{\tau} \in \mathcal{T}\}$ respectively. ▶

Example 2. For each string $u \in \Sigma^*$, there are infinitely many istructs in the string inductive structure of Example 1 corresponding to u. E.g., if $u = ab$, then istructs

$$\begin{gathered} \pi_a\langle 1,2,3\rangle \triangleleft \pi_b\langle 3,4,5\rangle \\ \pi_a\langle 2,3,4\rangle \triangleleft \pi_b\langle 4,5,6\rangle \\ \text{etc.} \end{gathered}$$

differing only in site labels, correspond to u.

However, for each string $u \in \Sigma^*$, there exists unique *struct* $\bar{\boldsymbol{\gamma}}_u$ corresponding to it, which, for convenience, we will denote identically ($\bar{\boldsymbol{\gamma}}_u = u$).

Similarly, every set of context-dependent insertion operations uniquely corresponds to a *transformation*. The transformation which corresponds to the insertion of string v between strings u and w will be denoted as $uw \to uvw$. $\diamondsuit$

Lemma 22. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be two istructs and let $h : \mathrm{ext}(\boldsymbol{\alpha}) \to \mathcal{S}$ be a site replacement such that $\boldsymbol{\beta} = \boldsymbol{\alpha}\langle h\rangle$. Then $\bar{\boldsymbol{\alpha}} = \bar{\boldsymbol{\beta}}$.

Proof. Let $\boldsymbol{\gamma} \in \bar{\boldsymbol{\beta}}$. Then $\boldsymbol{\gamma} = \boldsymbol{\beta}\langle f\rangle$ for some site replacement $f : \mathrm{ext}(\boldsymbol{\beta}) \to \mathcal{S}$. According to Lemma 10,

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha}\langle h\rangle)\langle f\rangle = \boldsymbol{\alpha}\langle f \circ h\rangle \in \bar{\boldsymbol{\alpha}}.$$

Therefore, $\bar{\boldsymbol{\beta}} \subseteq \bar{\boldsymbol{\alpha}}$.

By Lemma 11, there exists site replacement $h' : \mathrm{ext}(\boldsymbol{\beta}) \to \mathcal{S}$ such that $\boldsymbol{\alpha} = \boldsymbol{\beta}\langle h'\rangle$, so we have $\bar{\boldsymbol{\alpha}} \subseteq \bar{\boldsymbol{\beta}}$.

Hence, $\bar{\boldsymbol{\alpha}} = \bar{\boldsymbol{\beta}}$. ■

Definition 22. A finite set of transformations $T \subset \bar{\mathcal{T}}$ will be called a **transformation set**. ▶

As was mentioned above and as can be seen from the above example, it is structs that encode the real-world objects, since, if we used istructs or composites for this purpose, we would have had infinitely many encodings of the same object. However, now, having adopted this interpretation, we are faced with the situation in which it is impossible to define the action of a transformation on a struct. We address this problem by introducing, in the next section, the concepts of ipaths and paths induced by a transformation set. We will then define,in the following sections, operations of composition and embedding on paths via ipaths.

3.2 Ipaths and paths

We first introduce the concept of ipath.

Definition 23. Let T be a transformation set in $(\Pi, \mathcal{I})$. If $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_{n+1}$ are istructs and $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n$ are itransformations such that $\bar{\boldsymbol{\tau}}_1, \ldots, \bar{\boldsymbol{\tau}}_n \in T$ and

$$\boldsymbol{\alpha}_{i+1} = \boldsymbol{\alpha}_i \triangleleft \boldsymbol{\tau}_i, \quad i \in [1, n],$$

then the tuple

$$(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \boldsymbol{\alpha}_2, \boldsymbol{\tau}_2, \ldots, \boldsymbol{\tau}_n, \boldsymbol{\alpha}_{n+1})$$

will be called a **T-ipath** from $\boldsymbol{\alpha}_1$ to $\boldsymbol{\alpha}_{n+1}$, or simply ipath when no confusion arises. The set of all T-ipaths will be denoted by IP_T.

For an ipath c, $\text{begin}(c) \stackrel{\text{def}}{=} \boldsymbol{\alpha}_1$, $\text{end}(c) \stackrel{\text{def}}{=} \boldsymbol{\alpha}_{n+1}$ will be called the **beginning** and the **end** of ipath c, respectively.

The **length**, $|c|$, of the above ipath c is defined to be n:

$$|c| \stackrel{\text{def}}{=} n.$$

►

Definition 24. Ipaths $(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_m, \boldsymbol{\alpha}_{m+1}), (\boldsymbol{\beta}_1, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_n, \boldsymbol{\beta}_{n+1}) \in IP_T$ will be called **equivalent**, if $m = n$ and there exist site replacements $g_i : \text{ext}(\boldsymbol{\alpha}_i) \to \mathcal{S},\ i \in [1, n+1]$ and $h_i : \text{ext}(\boldsymbol{\tau}_j) \to \mathcal{S},\ i \in [1, n]$ such that

$$\begin{aligned} \boldsymbol{\beta}_i &= \boldsymbol{\alpha}_i\langle g_i\rangle, \quad i \in [1, n+1] \\ \boldsymbol{\delta}_i &= \boldsymbol{\tau}_i\langle h_i\rangle, \quad i \in [1, n] \end{aligned}$$

and for $i \in [1, n]$

$$\begin{aligned} h_i\big|_{\text{ext}(\boldsymbol{\tau}_i)\cap\text{ext}(\boldsymbol{\alpha}_i)} &= g_i\big|_{\text{ext}(\boldsymbol{\tau}_i)\cap\text{ext}(\boldsymbol{\alpha}_i)} \\ h_i\big|_{\text{ext}(\boldsymbol{\tau}_i)\cap\text{ext}(\boldsymbol{\alpha}_{i+1})} &= g_i\big|_{\text{ext}(\boldsymbol{\tau}_i)\cap\text{ext}(\boldsymbol{\alpha}_{i+1})}. \end{aligned}$$

We will denote the equivalence of ipaths $c_1, c_2 \in IP_T$ as $c_1 \sim c_2$. ►

Example 3. In SIS, let T be a transformation set consisting of one transformation $\bar{\boldsymbol{\tau}} = (\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\beta}})$, where $\boldsymbol{\beta} = [\pi_a\langle 1, 2, 3\rangle]$. This transformation corresponds to the context-independent insertion of letter a. Consider two T-ipaths:

$$\begin{aligned} c_1 &= (\pi_a\langle 1, 2, 3\rangle,\ \boldsymbol{\tau}\langle 2, 4, 5\rangle,\ \pi_a\langle 1, 2, 3\rangle \lhd \boldsymbol{\tau}\langle 2, 4, 5\rangle) \\ c_2 &= (\pi_a\langle 1, 2, 3\rangle,\ \boldsymbol{\tau}\langle 3, 4, 5\rangle,\ \pi_a\langle 1, 2, 3\rangle \lhd \boldsymbol{\tau}\langle 3, 4, 5\rangle). \end{aligned}$$

These ipaths correspond to the insertions of letter a at the beginning and at the end of string "a". They are *not* equivalent. Indeed, assume otherwise, and let h_1, g_1, h_2 be site replacements that establish the equivalence. By Def. 24, $h_1 = \text{id}$ and $g_1\big|_{\{2\}} = h_1\big|_{\{2\}}$, which implies that $\boldsymbol{\tau}\langle 3, 4, 5\rangle = \boldsymbol{\tau}\langle 2, 4, 5\rangle$, contradiction!

However, ipath

$$c_3 = (\pi_a\langle 2, 3, 4\rangle, \boldsymbol{\tau}\langle 3, 5, 6\rangle, \pi_a\langle 2, 3, 4\rangle \lhd \boldsymbol{\tau}\langle 3, 5, 6\rangle)$$

is equivalent to c_1. The corresponding site replacements are defined as follows:

$$\begin{aligned} g_1(i) &= i + 1, \quad i \in \{1, 2, 3\} \\ h_1(i) &= i + 1, \quad i \in \{2, 4, 5\} \\ g_2(i) &= i + 1, \quad i \in \{1, 3, 4, 5\}. \end{aligned}$$

♢

We now introduce the concept of path.

Definition 25. For an ipath c, the set

$$\bar{c} \stackrel{\text{def}}{=} \{c' \in IP_T \mid c' \sim c\}$$

will be called the ***T*-path**, or simply path, corresponding to c.

For a path $\bar{c}$, $\text{begin}(\bar{c}) \stackrel{\text{def}}{=} \text{begin}(c)$ and $\text{end}(\bar{c}) \stackrel{\text{def}}{=} \text{end}(c)$ will be called the **beginning** and the **end** of path $\bar{c}$, respectively; $|\bar{c}| \stackrel{\text{def}}{=} |c|$ is the **length of path** $\bar{c}$.

For $n \geq 0$, P_T^n will denote the set of all paths of length n, and $P_T \stackrel{\text{def}}{=} \cup_{n=0}^{\infty} P_T^n$ will denote the set of all paths.

For $\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}} \in \bar{\Theta}$,

$$P_T(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}) \stackrel{\text{def}}{=} \{p \in P_T \mid \text{begin}(p) = \bar{\boldsymbol{\alpha}},\ \text{end}(p) = \bar{\boldsymbol{\beta}}\}$$

is the set of all paths from $\bar{\boldsymbol{\alpha}}$ to $\bar{\boldsymbol{\beta}}$. Finally,

$$P_T^n(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}) \stackrel{\text{def}}{=} P_T(\bar{\boldsymbol{\alpha}}, \bar{\boldsymbol{\beta}}) \cap P_T^n.$$

►

Example 4. For the transformation set T from the previous example, there are two different paths between structs a and aa:

$$P_T(a, aa) = \{\bar{c}_1, \bar{c}_2\}.$$

♢

Lemma 23. For any path p, begin(p) and end(p) are correctly defined, i.e. they do not depend on the choice of ipath $c \in p$.

Proof. Let

$$c_1 = (\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\alpha}_{m+1})$$
$$c_2 = (\boldsymbol{\beta}_1, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\beta}_{n+1})$$

and $c_1, c_2 \in p$. By Def. 25, $c_1 \sim c_2$. Therefore, by Def. 24, in particular, there exists a site replacement $h_1 : \text{ext}(\boldsymbol{\alpha}_1) \to \mathcal{S}$ such that $\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_1\langle h_1 \rangle$. According to Lemma 22, $\bar{\boldsymbol{\beta}}_1 = \bar{\boldsymbol{\alpha}}_1$, i.e. begin$(p)$ does not depend on the choice of ipath $c \in p$.

Similarly, $m = n$, and $\bar{\boldsymbol{\beta}}_{n+1} = \bar{\boldsymbol{\alpha}}_{m+1}$. Hence, end$(p)$ does not depend on the choice of $c \in p$. ■

3.3 Path composition

Definition 26. If $c_1 = (\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n, \boldsymbol{\alpha}_{n+1})$ and $c_2 = (\boldsymbol{\alpha}_{n+1}, \boldsymbol{\tau}_{n+1}, \ldots, \boldsymbol{\tau}_{n+m}, \boldsymbol{\alpha}_{n+m+1})$ are T-ipaths, then T-ipath

$$c_2 \circ c_1 \stackrel{\text{def}}{=} (\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n+m}, \boldsymbol{\alpha}_{n+m+1})$$

will be called the **composition** of c_1 and c_2.

For convenience, we define $c_2 \circ c_1 = c_1$, if $\text{end}(c_1) \neq \text{begin}(c_2)$. ►

Definition 27. For two paths $p_1, p_2 \in P_T$ satisfying $\mathrm{end}(p_1) = \mathrm{begin}(p_2)$, the **composition** of p_1 and p_2 is defined as

$$p_2 \circ p_1 \stackrel{\text{def}}{=} \bar{c},$$

where $c = c_2 \circ c_1$, $c_1 \in p_1$, $c_2 \in p_2$, and $\mathrm{end}(c_1) = \mathrm{begin}(c_2)$.

Again, if $\mathrm{end}(p_1) \neq \mathrm{begin}(p_2)$, then $p_2 \circ p_1 \stackrel{\text{def}}{=} p_1$.

▶

Lemma 24. The composition of paths is defined correctly, i.e. it does not depend on the choice of ipaths c_1 and c_2.

Proof. Let c_1, c_2, c_1', c_2' be ipaths, such that $c_1 \sim c_1'$, $c_2 \sim c_2'$ and

$$\begin{aligned} \mathrm{end}(c_1) &= \mathrm{begin}(c_2) \\ \mathrm{end}(c_1') &= \mathrm{begin}(c_2'). \end{aligned} \tag{$*$}$$

We need to show that $c_2 \circ c_1 \sim c_2' \circ c_1'$.

By Def. 24, for $j \in \{1, 2\}$, we have

$$\begin{aligned} c_j &= (\boldsymbol{\alpha}_1^j, \boldsymbol{\tau}_1^j, \ldots, \boldsymbol{\alpha}_{n_j+1}^j) \\ c_j' &= (\boldsymbol{\beta}_1^j, \boldsymbol{\delta}_1^j, \ldots, \boldsymbol{\beta}_{n_j+1}^j), \end{aligned}$$

where

$$\boldsymbol{\alpha}_{n_1}^1 = \boldsymbol{\alpha}_1^2, \qquad \boldsymbol{\beta}_{n_1}^1 = \boldsymbol{\beta}_1^2,$$

and there exist site replacements

$$\begin{aligned} h_i^j &: \mathrm{ext}(\boldsymbol{\alpha}_i^j) \to \mathcal{S}, \qquad i \in [1, n_j] \\ g_i^j &: \mathrm{ext}(\boldsymbol{\tau}_i^j) \to \mathcal{S}, \qquad i \in [1, n_j], \end{aligned}$$

such that the conditions in Def. 24 are fulfilled. It follows from $(*)$ that one can choose these site replacements in such a way that $h_{n_1+1}^1 = h_1^2$ and $g_{n_1}^1 = g_1^2$. Therefore, the following site replacements are correctly specified:

$$\begin{aligned} h_i &= h_i^1, & i &\in [1, n_1+1] \\ h_i &= h_{i-n_1}^2, & i &\in [n_1+1, n_1+1+n_2] \\ g_i &= g_i^1, & i &\in [1, n_1] \\ g_i &= g_{i-n_1}^2, & i &\in [n_1+1, n_1+n_2]. \end{aligned}$$

This set of site replacements satisfies the conditions of Def. 24. Hence, $c_2 \circ c_1 \sim c_2' \circ c_1'$. ■

3.4 Elementary paths

In an inductive structure, the set of all "basic" (with respect to T) paths beginning at a struct $\bar{\boldsymbol{\gamma}}$ will play a useful role below.

Definition 28. Let, as above, T be a transformation set in inductive structure $(\Pi, \mathcal{I})$. For a struct $\bar{\boldsymbol{\gamma}} \in \bar{\Theta}$,

$$EP_T(\bar{\boldsymbol{\gamma}}) \stackrel{\text{def}}{=} \bigcup_{\bar{\boldsymbol{\alpha}} \in \bar{\Theta}} P^1_T(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\alpha}})$$

will be called the **set of elementary path from $\bar{\boldsymbol{\gamma}}$.**

For an elementary path $\bar{c} \in EP_T(\bar{\boldsymbol{\gamma}})$, where $c = (\boldsymbol{\gamma}, \boldsymbol{\tau}, \boldsymbol{\alpha})$, the transformation $\bar{\boldsymbol{\tau}} \in \bar{\mathcal{T}}$ will be called the **transformation for elementary path $\bar{c}$.** ▶

Lemma 25. For all $\bar{\boldsymbol{\gamma}} \in \bar{\Theta}$, $EP_T(\bar{\boldsymbol{\gamma}})$ is finite.

Proof. Since T is finite, it is sufficient to prove that for every $\bar{\boldsymbol{\tau}} \in T$, the set

$$\bigcup_{\bar{\boldsymbol{\gamma}}' \in \bar{\Theta}} P^1_{\{\bar{\boldsymbol{\tau}}\}}(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}')$$

is finite.

Suppose that on the contrary

$$\bigcup_{\bar{\boldsymbol{\gamma}}' \in \bar{\Theta}} P^1_{\{\bar{\boldsymbol{\tau}}\}}(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}') = \{p_i \,|\, i \in \mathbb{Z}_+\},$$

where $p_i \neq p_j$ if $i \neq j$.

Let $\boldsymbol{\gamma}, \boldsymbol{\tau}$ be some fixed elements from $\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\tau}}$, respectively.

For each $i \in \mathbb{Z}_+$, choose an ipath $c_i \in p_i$, $c_i = (\boldsymbol{\gamma}_i, \boldsymbol{\tau}_i, \boldsymbol{\gamma}'_i)$. There exist site replacements $f_i : \text{ext}(\boldsymbol{\gamma}_i) \to \mathcal{S}$, $g_i : \text{ext}(\boldsymbol{\tau}_i) \to \mathcal{S}$ such that $\boldsymbol{\gamma} = \boldsymbol{\gamma}_i\langle f_i \rangle$, $\boldsymbol{\tau} = \boldsymbol{\tau}_i\langle g_i \rangle$.

Let $h_i : \text{ext}(\boldsymbol{\gamma}_i) \to \mathcal{S}$ be a site replacement defined as

$$h_i(s) = \begin{cases} g_i(s), & s \in \text{ext}(\boldsymbol{\tau}_i) \\ f_i(s), & s \notin \text{ext}(\boldsymbol{\tau}_i). \end{cases}$$

Let

$$\begin{aligned} \boldsymbol{\delta}_i &= \boldsymbol{\gamma}_i\langle h_i \rangle \\ \boldsymbol{\delta}'_i &= \boldsymbol{\delta}_i \lhd \boldsymbol{\tau}. \end{aligned}$$

We have $c'_i = (\boldsymbol{\delta}_i, \boldsymbol{\tau}, \boldsymbol{\delta}'_i) \sim c_i$. Moreover, $\text{ext}(\boldsymbol{\delta}_i) \subseteq \text{ext}(\boldsymbol{\gamma}) \cup \text{ext}(\boldsymbol{\tau})$. By Lemma 13, there exist indices i, j, $i \neq j$, such that $\boldsymbol{\delta}_i = \boldsymbol{\delta}_j$. Hence, $\boldsymbol{\delta}'_i = \boldsymbol{\delta}'_j$, $c'_i = c'_j$ and $p_i = p_j$, contradiction! ■

3.5 Path embedding

Since in an inductive structure its T-path are of great importance, in this section, we introduce in the simplest but important relationship between two path.

Definition 29. For two T-ipaths $c_1 = (\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_m, \boldsymbol{\alpha}_{m+1})$ and $c_2 = (\boldsymbol{\beta}_1, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_m, \boldsymbol{\beta}_{m+1})$, we will say that c_1 **can be embedded in** c_2 and denote this fact by $c_1 \hookrightarrow c_2$, if $m = n$ and there exists istruct $\boldsymbol{\gamma}$ such that $\boldsymbol{\beta}_1 = \boldsymbol{\gamma} \lhd \boldsymbol{\alpha}_1$ and for all $i \in [1, n]$, $\boldsymbol{\delta}_i = \boldsymbol{\tau}_i$.

A **path** p_1 **can be embedded in path** p_2 $(p_1 \hookrightarrow p_2)$, if there exist ipaths $c_1 \in p_1$, $c_2 \in p_2$, such that c_1 can be embedded in c_2. ▶

Example 5. Continuing Example 3, let two ipath be

$$c_6 = (\pi_a\langle 2,4,5\rangle,\ \boldsymbol{\tau}\langle 4,6,7\rangle,\ \pi_a\langle 2,4,5\rangle \lhd \boldsymbol{\tau}\langle 4,6,7\rangle)$$
$$c_7 = (\pi_a\langle 1,2,3\rangle \lhd \boldsymbol{\tau}\langle 2,4,5\rangle,\ \boldsymbol{\tau}\langle 4,6,7\rangle,\ \pi_a\langle 1,2,3\rangle \lhd \boldsymbol{\tau}\langle 2,4,5\rangle \lhd \boldsymbol{\tau}\langle 4,6,7\rangle).$$

Then, $c_6 \hookrightarrow c_7$, and the corresponding $\boldsymbol{\gamma}$ from Def. 29 is equal to $[\pi_a\langle 1,2,3\rangle]$.

Path $\bar{c}_6$ corresponds to the insertion of letter a at the beginning of string "a", and path $\bar{c}_7$ corresponds to the insertion of letter a in the middle of string "aa".

For ipath

$$c_9 = (\pi_a\langle 1,2,3\rangle \lhd \boldsymbol{\tau}\langle 2,4,5\rangle,\ \boldsymbol{\tau}\langle 5,6,7\rangle,\ \pi_a\langle 1,2,3\rangle \lhd \boldsymbol{\tau}\langle 2,4,5\rangle \lhd \boldsymbol{\tau}\langle 5,6,7\rangle),$$

which represents insertion of a at the end of "aa", $c_6 \not\hookrightarrow c_9$, and $\bar{c}_6 \not\hookrightarrow \bar{c}_9$. ◊

Lemma 26. If $c_1 \hookrightarrow c_2$, then for all $i \in [1, m+1]$, $\boldsymbol{\beta}_i = \boldsymbol{\gamma} \lhd \boldsymbol{\alpha}_i$.

Proof. By Def. 29, $\boldsymbol{\beta}_1 = \boldsymbol{\gamma} \lhd \boldsymbol{\alpha}_1$. Let $i \in [1,m]$ and $\boldsymbol{\beta}_i = \boldsymbol{\gamma} \lhd \boldsymbol{\alpha}_i$. Then,

$$\boldsymbol{\beta}_{i+1} = \boldsymbol{\beta}_i \lhd \boldsymbol{\tau}_i = (\boldsymbol{\gamma} \lhd \boldsymbol{\alpha}_i) \lhd \boldsymbol{\tau}_i \overset{\text{L. 21}}{=} \boldsymbol{\gamma} \lhd (\boldsymbol{\alpha}_i \lhd \boldsymbol{\tau}_i) = \boldsymbol{\gamma} \lhd \boldsymbol{\alpha}_{i+1},$$

and the Lemma is proved by induction on i. ■

Lemma 27. If $p_1 \hookrightarrow p_2$, then for all $c_1 \in p_1$ there exists $c_2 \in p_2$ such that $c_1 \hookrightarrow c_2$.

Proof. Let $p_1 \hookrightarrow p_2$. By Def. 29, there exist ipaths $c_1^0 \in p_1$, $c_2^0 \in p_2$ such that $c_1^0 \hookrightarrow c_2^0$.

Let

$$\begin{aligned} c_1 &= (\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\alpha}_{m+1}) \in p_1 \\ c_1^0 &= (\boldsymbol{\alpha}_1^0, \boldsymbol{\tau}_1^0, \ldots, \boldsymbol{\alpha}_{m_0+1}^0) \\ c_2^0 &= (\boldsymbol{\beta}_1^0, \boldsymbol{\tau}_1^0, \ldots, \boldsymbol{\beta}_{m_0+1}^0). \end{aligned}$$

Since $c_1 \sim c_1^0$, $m = m_0$ and, by Def. 24 there exist site replacements $g_i : \text{ext}(\boldsymbol{\alpha}_i^0) \to \mathcal{S}$, $h_i : \text{ext}(\boldsymbol{\tau}_i^0) \to \mathcal{S}$.

Let $g_i' : \text{ext}(\boldsymbol{\beta}_i^0) \to \mathcal{S}$ be any site replacements satisfying

$$g_i'\big|_{\text{ext}(\boldsymbol{\alpha}_i^0)} = g_i,$$

and, for convenience, let $h_i' = h_i$. Then

$$c_2 = (\boldsymbol{\beta}_1^0\langle g_1'\rangle, \boldsymbol{\tau}_1^0\langle h_1'\rangle, \ldots, \boldsymbol{\beta}_{m+1}^0\langle g_{m+1}'\rangle) \sim c_2^0,$$

and $c_1 \hookrightarrow c_2$. ■

3.6 Generating process, typicality measure, and class

As we have seen in the previous sections, a transformation set induces a set of paths between structs. From now on, we will interpret them as *struct generation* paths and will introduce in this section the corresponding generating process. For transformation set itself, we introduce a numeric parameter for each transformation (which controls the flow of the process in time) and a distinct struct with which the process starts.

Definition 30. A pair $WT = (T, l)$, where T is a transformation set and $l : T \to \mathbb{R}_+$ is a mapping, will be called a **weighted transformation set**. Perhaps, the simplest but not neccessesarily the best interpretation of the transformation weight $l(\bar{\boldsymbol{\tau}})$ is the "average time it takes for the transformation $\bar{\boldsymbol{\tau}} \in T$ to be completed".

A triple $\mathbf{TS} = (T, l, \bar{\boldsymbol{\kappa}})$, where (T, l) is a weighted transformation set and $\bar{\boldsymbol{\kappa}} \in \bar{\Theta}$ is a struct called the **progenitor**, will be called a **transformation system**. ▶

Definition 31. For a transformation system $\mathbf{TS} = (T, l, \bar{\boldsymbol{\kappa}})$, $T = \{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \ldots, \boldsymbol{\tau}_m\}$, we will call set

$$TS \stackrel{\text{def}}{=} \{\bar{\gamma} \in \bar{\Theta} \mid P_T(\bar{\boldsymbol{\kappa}}, \bar{\gamma}) \neq \varnothing\}$$

the **set of structs generated by TS**. ▶

Definition 32. Let $\mathbf{TS} = (T, l, \bar{\boldsymbol{\kappa}})$ be a transformation system in inductive structure $(\Pi, \mathcal{I})$, and let

$$c = (\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n, \boldsymbol{\alpha}_{n+1}) \in IP_T.$$

The **duration of ipath** c (and also of **path** $\bar{c}$) is defined as

$$l(\bar{c}) = l(c) \stackrel{\text{def}}{=} l(\boldsymbol{\tau}_1) + \ldots + l(\boldsymbol{\tau}_n).$$

▶

The motivation for the following, one of the central concepts is given right after its definition. In short, for a transformation system, we need the process that actually constructs the structs generated by the transformation system. We want to emphasize that we do not insist on this particular way of "unfolding" (in time) the information provided by the transformation system, i.e. via the stochastic process, and, in fact, in section Future work are encouraging others to look for different/interesting approaches to the generation of the class elements, not relying on stochastic processes.

Definition 33. For a transformation system $\mathbf{TS} = (T, l, \bar{\boldsymbol{\kappa}})$, the **generating process** $G_{\mathbf{TS}}$ (or simply G) is a countable state Markov stochastic process defined as follows:[14]

1. The states of G are elements of the set TS of structs generated by **TS**.
2. The amount of time which G spends in state $\bar{\gamma}$ is a random variable distributed exponentially with mean

$$L = \frac{1}{\sum\limits_{p \in EP_T(\bar{\gamma})} 1/l(p)}\,. \qquad (*)$$

3. When G leaves state $\bar{\gamma}$, it chooses randomly an elementary path $p \in EP_T(\bar{\gamma})$ with probability

$$\frac{L}{l(p)} \qquad (**)$$

and enters state $\text{end}(p)$.

[14] We intentionally use "physical" terminology referring to the "states of the system".

4. All random variables in 2 and 3 are mutually independent.

▶

Remark 2. We view the process of struct generation (by a transformation system) as a stochastic process, since the choice of the next elementary path and the time it takes is variable (in nature), and, in general, depends not only on the transformation system itself but also on the external factors the influence of which cannot be taken into account.

The choice by the process of the next state is modeled based on the following assumptions. First, all elementary paths are indivisible. I.e., an elementary path is either "completed" or not "completed", since there are no "intermediate" elementary paths. Therefore, the expected amount of time for an elementary path to be completed is independent of the time one has already been waiting for the completion of the elementary path. And this property gives us the exponential distribution for the amount of time which the process spends in each state. Second, the elements of $EP_T(\bar{\gamma})$ are all, *independently* of each other and simultaneously, "competing" for the application but only one of them is actually completed first. These two assumptions lead to formulas $(*)$ and $(**)$.

We assume all random variables to be independent, since we postulate that the choice of the transformation that is applied to a particular struct and the time of its application depend only on the struct's structure. I.e., these random variables depend neither on the path that has lead to the struct, nor on the time it took to generate the struct.

Definition 34. Let **TS** be a transformation system and G be the generating process for **TS**. Let $E_G(\bar{\gamma})$ be the expected time spent by G in state $\bar{\gamma}$. We define

$$\bar{E}_G \stackrel{\text{def}}{=} \sum_{\bar{\gamma} \in TS} E_G(\bar{\gamma}).$$

If $\bar{E}_G$ is finite, we will say that transformation system **TS** satisfies the **typicality measure existence condition** and we will call such a transformation system a **class transformation system** or simply **class**. The set of elements of class **C** will be denoted by C. ▶

The next definition introduces an important concept of struct typicality with respect to a class **C**.

Definition 35. Let **C** be a class and G be the generating process for **C**. For $\bar{\gamma} \in C$, we define

$$\nu_{\mathbf{C}}(\bar{\gamma}) \stackrel{\text{def}}{=} \frac{E_G(\bar{\gamma})}{\bar{E}_G}.$$

For $\bar{\gamma} \notin C$, $\nu_C(\bar{\gamma}) \stackrel{\text{def}}{=} 0$. Measure $\nu_{\mathbf{C}}$ on $\bar{\Theta}$ will be called **C-typicality measure**, or simply the **typicality**. ▶

Remark 3. For a transformation system $\mathbf{TS} = (T, l, \bar{\boldsymbol{\kappa}})$ and a path $p \in P_T$, we will use the phrase **"process G passes (entire) p"** to refer to an intuitively clear random event, and the probability of this event is specified uniquely by the process G. Then, the duration of path p is equal to the expected time the generating process spends on path p under the condition that it passes p.

Definition 36. For a class $\mathbf{C} = \mathbf{TS}$ and path $p \in P_{\mathbf{C}} \stackrel{\text{def}}{=} P_{\mathbf{TS}}$, the **probability of** p is defined as

$$\mu_C(p) \stackrel{\text{def}}{=} P(G \text{ passes any path } p' \text{ into which } p \text{ can be embedded}).$$

►

Example 6. Continuing Examples 1, 2, consider the following class (specified by its transformation system)

$$\mathbf{C} = (\{aa \to aba, ab \to acb, ab \to adb\}, (l_1, l_2, l_3), aa),$$

and let $G = G_{\mathbf{C}}$ be the generating process for $\mathbf{C}$.

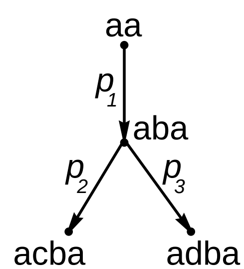

Figure 10: Elements and paths of class **C** in Example 2.

Paths of class **C** reachable by the generating process are shown in Fig. 10 and have the following probabilities:

$$\begin{aligned}
&\mu_{\mathbf{C}}(p_1) = 1\\
&\mu_{\mathbf{C}}(p_2) = \mu_{\mathbf{C}}(p_2 \circ p_1) = \tfrac{1/l_2}{1/l_2+1/l_3}\\
&\mu_{\mathbf{C}}(p_3) = \mu_{\mathbf{C}}(p_3 \circ p_1) = \tfrac{1/l_3}{1/l_2+1/l_3}.
\end{aligned}$$

The set of class elements is $C = \{aa, aba, acba, adba\}$. The expected times the generating process spends in each class element are:

$$\begin{aligned}
&E_G(aa) = l_1\\
&E_G(aba) = \tfrac{1}{1/l_2+1/l_3}\\
&E_G(acba) = E_G(adba) = 0.
\end{aligned}$$

Therefore,

$$\bar{E}_G = l_1 + \frac{1}{1/l_2 + 1/l_3},$$

and

$$\begin{aligned}
&\nu_{\mathbf{C}}(aa) = l_1/\bar{E}_G\\
&\nu_{\mathbf{C}}(aba) = \tfrac{1}{\bar{E}_G(1/l_2+1/l_3)}\\
&\nu_{\mathbf{C}}(acba) = \nu_{\mathbf{C}}(adba) = 0.
\end{aligned}$$

◊

Remark 4. Note that the typicality of each "terminal" class element is always zero.

3.7 Extended string inductive structure and the class of strings

In this section we define a new inductive structure—extended string inductive structure (ESIS)—and the class of strings. One should note the differences between the class of strings introduced next and the string inductive structure (SIS) introduced in Example 1 (section 2.4) . First, elements of the class of strings correspond to strings only while some structs in SIS correspond to finite sets of strings. The desire to limit oneself to structs which represent single strings only can now be realized using context-dependent itransformations. Second, a typicality measure is defined for elements of the string class, and we will show how to compute the typicality of any class element. We introduce two new primitives, σ and π_x, to construct itransformation contexts and "termination" transformations, which will ensure the existence of the typicality measure. For simplicity, we consider strings over the alphabet $\Sigma = \{a, b\}$. It is also important to keep in mind that the inductive structure introduced here is one of the *simplest* inductive structures for capturing a very popular but apparently *incompletely specified* "string" representation.

The set of abstract sites for the new inductive structure is $A = \{a_1, a_2, a_3\}$ and the set of concrete sites is $\mathcal{S} = \{0, 1, 2, \ldots\}$.

The set of primtypes is $\Pi_{ESIS} = \{\sigma, \pi_a, \pi_b, \pi_x\}$ (Fig. 11), where

$$\begin{array}{ll} \text{init}(\sigma) = \{a_1\} & \text{term}(\sigma) = \{a_2\} \\ \text{init}(\pi_x) = \{a_1\} & \text{term}(\pi_x) = \{a_2\} \\ \text{init}(\pi_a) = \{a_1\} & \text{term}(\pi_a) = \{a_2, a_3\} \\ \text{init}(\pi_b) = \{a_1\} & \text{term}(\pi_b) = \{a_2, a_3\}. \end{array}$$

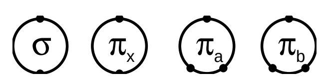

Figure 11: Primitive types of the ESIS.

The set of semantic identities of the ESIS is $\mathcal{I}_{ESIS} = \{\alpha_{ij} \equiv \beta_{ij} \,|\, i, j \in \Sigma\}$ (see Fig. 12), where

$$\begin{array}{rcl} \alpha_{ij} & = & \pi_i\langle 1, 2, 3\rangle \triangleleft \sigma\langle 2, 4\rangle \triangleleft \pi_j\langle 4, 5, 6\rangle \\ \beta_{ij} & = & \pi_j\langle 1, 5, 2\rangle \triangleleft \sigma\langle 2, 4\rangle \triangleleft \pi_i\langle 4, 6, 3\rangle. \end{array}$$

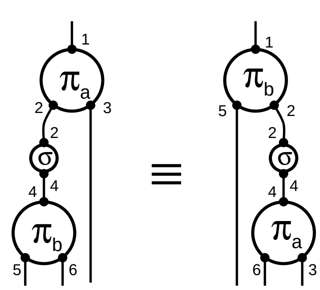

Figure 12: Semantic identities of the ESIS.

Since in each identity the lengths of the left- and the right-hand sides are equal, the induction axiom holds for the set of istructs Θ in $(\Pi_{ESIS}, \mathcal{I}_{ESIS})$.

As mentioned above, the resulting inductive structure $(\Pi, \mathcal{I})$ will be called the **extended string inductive structure**.

We now define a class $\mathbf{C} = (T, l, \bar{\boldsymbol{\kappa}})$ in inductive structure $(\Pi_{ESIS}, \mathcal{I}_{ESIS})$ called the **class of strings**. Recalling that for a transformation $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, $\text{ext}(\boldsymbol{\tau}) \stackrel{\text{def}}{=} \text{ext}(\boldsymbol{\alpha}) \cup \text{ext}(\boldsymbol{\beta})$, we define the transformation set to be $T = \{\bar{\boldsymbol{\tau}}_a, \bar{\boldsymbol{\tau}}_b, \bar{\boldsymbol{\tau}}_x, \bar{\boldsymbol{\tau}}_{xx}\}$ (see Fig. 13), where $\boldsymbol{\tau}_i = ([\alpha_i], [\beta_i])$ for $i \in \{a, b, x, xx\}$ and

$$\alpha_a = \sigma\langle 1,2\rangle$$
$$\beta_a = \sigma\langle 1,2\rangle \lhd \pi_a\langle 2,5,6\rangle \lhd \sigma\langle 5,3\rangle \lhd \sigma\langle 6,4\rangle$$

$$\alpha_b = \sigma\langle 1,2\rangle$$
$$\beta_b = \sigma\langle 1,2\rangle \lhd \pi_b\langle 2,5,6\rangle \lhd \sigma\langle 5,3\rangle \lhd \sigma\langle 6,4\rangle$$

$$\alpha_x = \sigma\langle 1,2\rangle$$
$$\beta_x = \sigma\langle 1,2\rangle \lhd \pi_x\langle 2,3\rangle$$

$$\alpha_{xx} = \sigma\langle 1,2\rangle \lhd \pi_x\langle 3,4\rangle$$
$$\beta_{xx} = \sigma\langle 1,2\rangle \lhd \pi_x\langle 3,4\rangle \lhd \pi_x\langle 2,5\rangle.$$

The weights of the transformations are defined as follows

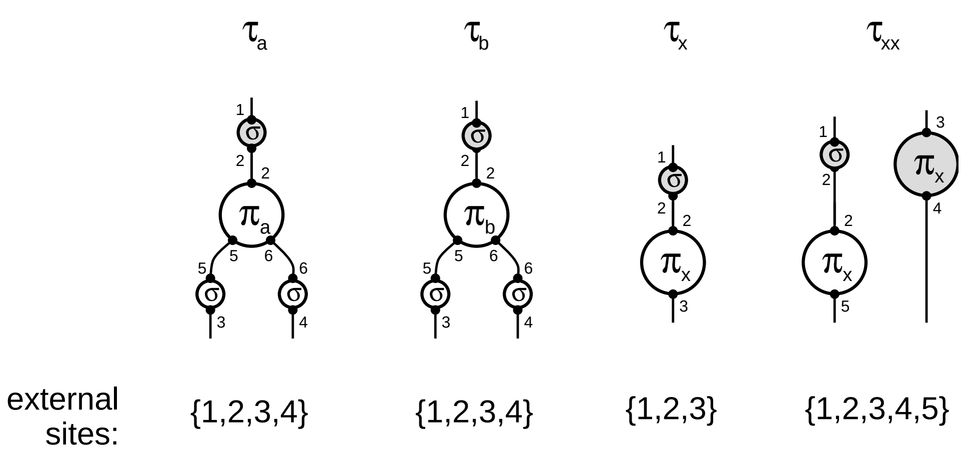

Figure 13: Transformations from the transformation set for the class of strings.

$$l(\bar{\boldsymbol{\tau}}_a) = l(\bar{\boldsymbol{\tau}}_b) = l(\bar{\boldsymbol{\tau}}_x) = 1,\ l(\bar{\boldsymbol{\tau}}_{xx}) = 0,$$

and the progenitor is $\boldsymbol{\kappa} = [\sigma\langle 1,2\rangle]$.

For each string $u = a_1 \dots a_n \in \Sigma^*$, there is a corresponding class element $\bar{\gamma}_u \in C$ (see Fig. 14), where [15]

$$\boldsymbol{\gamma}_u = \boldsymbol{\kappa} \lhd \boldsymbol{\tau}_{a_1}\langle 1,2,3,4\rangle \lhd \boldsymbol{\tau}_{a_2}\langle 0,4,5,6\rangle \lhd \boldsymbol{\tau}_{a_3}\langle 0,6,7,8\rangle \lhd \ldots \lhd \boldsymbol{\tau}_{a_n}\langle 0,2n,2n+1,2n+2\rangle. \quad (*)$$

Vice versa, if we take $\bar{\boldsymbol{\gamma}} \in C$, such that every non-empty path $p \in P_{\mathbf{C}}(\bar{\boldsymbol{\kappa}}, \bar{\boldsymbol{\gamma}})$ consists of elementary paths whose transformations are $\bar{\boldsymbol{\tau}}_a$ or $\bar{\boldsymbol{\tau}}_b$, then there exists a composite $\gamma \in \boldsymbol{\gamma}$ of the form $(*)$. By definition, string

$$\mathrm{str}(\bar{\boldsymbol{\gamma}}) \stackrel{\mathrm{def}}{=} a_1 \ldots a_n$$

is the string corresponding to struct $\bar{\boldsymbol{\gamma}}$. We will call such struct $\bar{\boldsymbol{\gamma}} \in C$ a **string-like struct**.

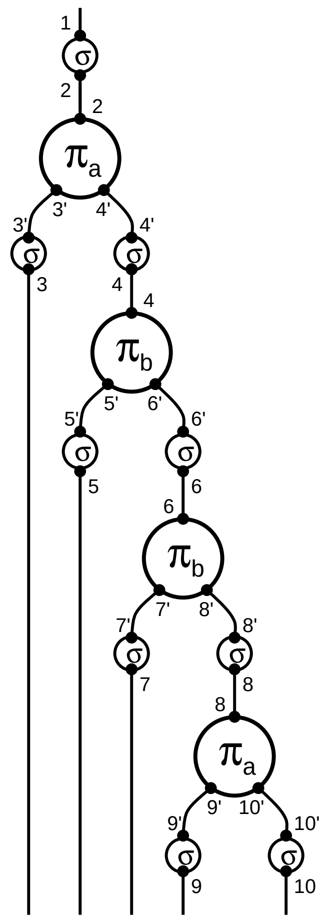

Figure 14: Element of the class of strings corresponding to string *abba*.

Lemma 28. The following statements are true for the above transformation system $\mathbf{C} = (T, l, \bar{\boldsymbol{\kappa}})$.

1. Transformation system $\mathbf{C}$ satisfies the typicality measure existence condition.

[15] For convenience and in view of Remark 1 (after Def. 16), we use here zero as the first value for some site replacements.

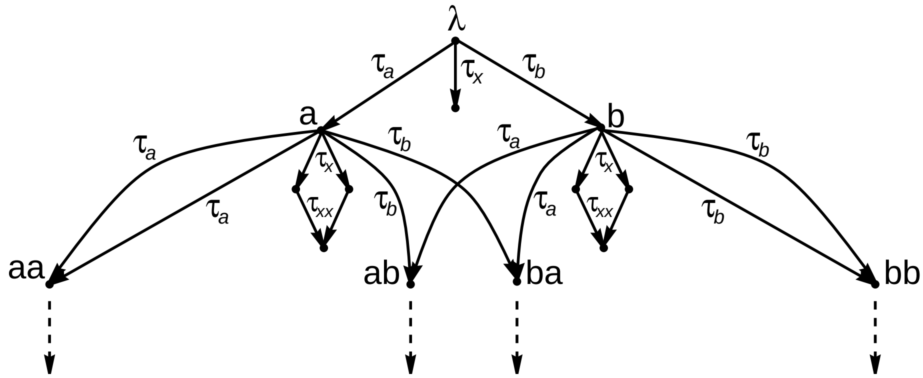

Figure 15: Paths and elements of the class of strings. String-like structs are labeled by the corresponding strings, and elementary paths are labeled by their transformations.

2. For $\bar{\gamma} \in C$,

$$\nu_{\mathbf{C}}(\bar{\gamma}) = \begin{cases} \frac{2}{3^{n+1}(n+1)\ln 3}\,, & \text{if } \bar{\gamma} \text{ is a string-like struct corresponding to a string of length } n \\ 0\,, & \text{otherwise.} \end{cases}$$

Proof. Take $\bar{\gamma} \in C$.

If $\bar{\gamma}$ is not a string-like struct, then either there exists an elementary path whose transformation is $\bar{\boldsymbol{\tau}}_{xx}$ from $EP_{\mathbf{C}}(\bar{\gamma})$, which the generating process choses with probability 1 (since $l(\bar{\boldsymbol{\tau}}_{xx}) = 0$), or $EP_{\mathbf{C}}(\bar{\gamma}) = \varnothing$. In either case, since $l(\bar{\boldsymbol{\tau}}_{xx}) = 0$, the expected time spent by the generating process in $\bar{\gamma}$ is 0.

If $\bar{\gamma}$ is a string-like struct corresponding to a string of length n, then the probability that the generating process passes a particular path $p \in P_{\mathbf{C}}(\bar{\boldsymbol{\kappa}}, \bar{\gamma})$ is equal to

$$\frac{1}{3} \cdot \frac{1}{6} \cdot \ldots \cdot \frac{1}{3n} = \frac{1}{3^n n!}\,.$$

There are $n!$ paths in $P_{\mathbf{C}}(\bar{\boldsymbol{\kappa}}, \bar{\gamma})$ (see Fig. 15). Hence, the probability that process G reaches $\bar{\gamma}$ is $1/3^n$. Since for all elementary paths $s \in EP_{\mathbf{C}}(\bar{\gamma})$, $l(s) = 1$, and there are $3(n+1)$ of them, the expected time spent by the process in $\bar{\gamma}$ is equal to $1/(3^n \cdot 3(n+1))$, i.e. $E_G(\bar{\gamma}) = 1/(3^{n+1}(n+1))$.

Since the string-like structs and strings are in one-to-one correspondence,

$$\bar{E}_G = \sum_{u \in \{a,b\}^*} E_G(\bar{\gamma}_u) = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}(n+1)} = \frac{\ln 3}{2}.$$

Thus, $\mathbf{C}$ satisfies the typicality measure existence condition, and

$$\nu_{\mathbf{C}}(\bar{\gamma}) = \frac{E_G(\bar{\gamma})}{\bar{E}_G} = \frac{2}{3^{n+1}(n+1)\ln 3}.$$

■

We will denote the constructed class of strings $\mathbf{C}$ in ESIS as $\mathbf{C}_{str}$.

3.8 Natural numbers inductive structure and the class of natural numbers

For this inductive structure, the set of abstract sites is $A = \{a_1, a_2\}$ and the set of concrete sites is $\mathcal{S} = \{0, 1, 2, \ldots\}$. The set of primtypes for the natural numbers inductive structure is $\Pi_{\mathbb{N}} = \{\pi_1, \pi_x\}$, where

$$\begin{aligned} &\mathrm{init}(\pi_1) = \mathrm{init}(\pi_x) = \{a_1\} \\ &\mathrm{term}(\pi_1) = \{a_2\} \\ &\mathrm{term}(\pi_x) = \varnothing \end{aligned}$$

The set of semantic identities is $\mathcal{I}_{\mathbb{N}} = \mathrm{Comm}(\Pi_{\mathbb{N}})$, i.e. the set of commutativity identities (see Def. 13).

The transformation set for the class of natural numbers is $T = \{\bar{\boldsymbol{\tau}}_1, \bar{\boldsymbol{\tau}}_x\}$, where

$$\begin{aligned} \boldsymbol{\tau}_1 &= ([\pi_1\langle 1,2\rangle],\ [\pi_1\langle 1,2\rangle \lhd \pi_1\langle 2,3\rangle]) \\ \boldsymbol{\tau}_x &= ([\pi_1\langle 1,2\rangle],\ [\pi_1\langle 1,2\rangle \lhd \pi_x\langle 2\rangle]). \end{aligned}$$

The transformation weights are defined as follows

$$l(\bar{\boldsymbol{\tau}}_1) = l(\bar{\boldsymbol{\tau}}_x) = 1,$$

and the progenitor is $\boldsymbol{\kappa} = [\pi_1\langle 1,2\rangle]$. We will call $\mathbf{C}_{\mathbb{N}} = (T, l, \bar{\boldsymbol{\kappa}})$ the transformation system for natural numbers.

For each number $n \in \mathbb{N}$, there is a corresponding struct $\bar{\gamma}_n \in C_{\mathbb{N}}$, where

$$\boldsymbol{\gamma}_n = \boldsymbol{\kappa} \lhd \boldsymbol{\tau}_1\langle 2,3\rangle \lhd \ldots \lhd \boldsymbol{\tau}_1\langle n, n+1\rangle. \qquad (*)$$

Such structs will be called **n**umber-like.

The probability that generating process $G_{\mathbb{N}}$ for transformation system $\mathbf{C}_{\mathbb{N}}$ will reach number-like element $\boldsymbol{\gamma}_n$ equals to $(1/2)^n$. Therefore,

$$\nu_{\mathbf{C}_{\mathbb{N}}}(\bar{\gamma}_n) = (1/2)^{n+1},$$

and the process termination condition is fulfilled, hence $\mathbf{C}_{\mathbb{N}}$ is a class.

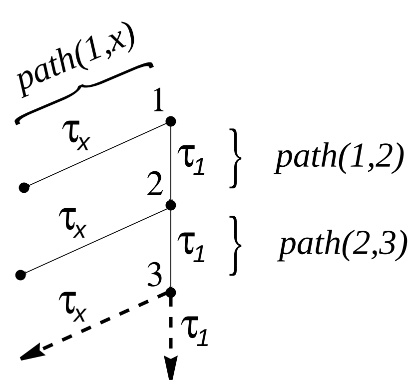

Figure 16: Paths and elements of the class of natural numbers.

Class $\mathbf{C}_{\mathbb{N}}$ satisfies the following properties (see also Fig. 16):

1. For every pair of natural numbers (m, n), $m < n$, there exists unique path in $P_{\mathbf{C}_{\mathbb{N}}}$ between $\bar{\boldsymbol{\gamma}}_m$ and $\bar{\boldsymbol{\gamma}}_n$, which we denote by $\text{path}(m, n)$.

2. For every natural number n, there exists unique elementary path in $EP_{\mathbf{C}_{\mathbb{N}}}$ beginning at $\bar{\boldsymbol{\gamma}}_n$ whose transformation is $\bar{\boldsymbol{\tau}}_x$. We denote this path by $\text{path}(n, x)$.

3. For all natural numbers m, n, k,

$$\begin{aligned}
&\text{path}(m, n) \hookrightarrow \text{path}(k + m, k + n) \\
&\text{path}(n, x) \hookrightarrow \text{path}(n + k, x) \\
&\text{path}(n, x) \circ \text{path}(m, n) \hookrightarrow \text{path}(n + k, x) \circ \text{path}(m + k, n + k).
\end{aligned}$$

3.9 Category of classes and class morphisms

In this and the next sections, we formalize the intuitive understanding of the relationship between the classes of (possibly different) inductive structures by introducing the concept of *morphism* [25]. In mathematics, a morphism between two analogous/homogeneous "mathematical structures" is a mapping which "carries" each operation in the domain into the corresponding operation in the codomain. Since, in this paper, our immediate goal is inductive learning of classes, the "mathematical structures" of interest are classes, and, therefore, the "structural elements" are paths. Earlier, we have defined three operations on paths: composition, embedding, and path duration.

We verify that the set of all classes (in all inductive structures) and morphisms between them form a category. The concepts of monomorphism (in the category of sets these correspond to injective mappings) and epimorphism (those corresponding to surjective mappings), as well as those of a subclass and a quotient-class follow from the definition of morphism in a standard manner. However, in view of its immediate utility, we will give, in the next (very short) section, explicit definition of a monomorphism.

Definition 37. Let $\mathbf{C}_1 = (T_1, l_1, \bar{\boldsymbol{\kappa}}_1)$, $\mathbf{C}_2 = (T_2, l_2, \bar{\boldsymbol{\kappa}}_2)$ be classes from inductive structures $(\Pi_1, \mathcal{I}_1), (\Pi_2, \mathcal{I}_2)$ respectively. A **morphism f from $\mathbf{C}_1$ to $\mathbf{C}_2$**, denoted

$$\mathbf{f} : \mathbf{C}_1 \to \mathbf{C}_2,$$

is *specified* by a mapping $f : P_{T_1} \to P_{T_2}$ satisfying for all $p_1, p_2, p \in P_{T_1}$ the following conditions:

1. $f(p_1 \circ p_2) = f(p_1) \circ f(p_2)$.

2. If $p_1 \hookrightarrow p_2$, then $f(p_1) \hookrightarrow f(p_2)$.

3. $l_2(f(p)) = l_1(p)$.

Two **morphisms**, $\mathbf{f}_1 : \mathbf{C}_1 \to \mathbf{C}_2$ and $\mathbf{f}_2 : \mathbf{C}_1 \to \mathbf{C}_2$, **are equal**, if the corresponding mappings f_1 and f_2 coincide on every path that the generating process G_1 for $\mathbf{C}_1$ can pass [16]. ▶

Example 7. We now define a morphism $\mathbf{f} : \mathbf{C}_{\mathbb{N}} \to \mathbf{C}_{str}$. Due to property 3 (applied to $\mathbf{C}_{\mathbb{N}}$), it is sufficient to define the mapping f on paths $\text{path}(1, 2)$, $\text{path}(2, 3)$, $\text{path}(1, x)$ and $\text{path}(2, x)$. Indeed, let the values of mapping f on these paths be fixed. We, first, prove that $f(\text{path}(3, 4))$ is uniquely defined.

[16] See Remark 3 (section 3.6).

- Since $\text{path}(1,2) \hookrightarrow \text{path}(2,3) \hookrightarrow \text{path}(3,4)$,

$$f(\text{path}(1,2)) \hookrightarrow f(\text{path}(3,4)) \hookrightarrow f(\text{path}(3,4)).$$

- Since $\text{path}(1,3) \hookrightarrow \text{path}(2,4)$, we have

$$f(\text{path}(2,3)) \circ f(\text{path}(1,2)) = f(\text{path}(2,3) \circ \text{path}(1,2)) = f(\text{path}(1,3)) \hookrightarrow \\ f(\text{path}(2,4)) = f(\text{path}(3,4) \circ \text{path}(2,3)) = f(\text{path}(3,4)) \circ f(\text{path}(2,3)).$$

The statement follows directly from the above two facts.

Similarly, we see that $f(\text{path}(n, n+1))$ is uniquely defined for all $n \in \mathbb{N}$.

Next, for all $n \in \mathbb{N}$,

$$\text{path}(2,x) \circ \text{path}(1,2) \hookrightarrow \text{path}(n+1,x) \circ \text{path}(n,n+1),$$

therefore $f(\text{path}(n+1,x))$ is uniquely defined.

To complete the proof, we note that since the values of f are uniquely defined on all elementary paths, f itself is uniquely defined.

Obviously, the values $f(\text{path}(1,2))$, $f(\text{path}(2,3))$, $f(\text{path}(1,x))$ and $f(\text{path}(2,x))$ can be specified in infinitely many ways. Let us look at one of them:

$$\begin{aligned}
&f_1(\text{path}(1,2)) = [\,(\boldsymbol{\kappa}_{str},\, \boldsymbol{\tau}_a\langle 1,2,3,4\rangle,\, \boldsymbol{\kappa}_{str} \lhd \boldsymbol{\tau}_a\langle 1,2,3,4\rangle)\,]\\
&f_1(\text{path}(2,3)) = [\,(\boldsymbol{\kappa}_{str} \lhd \boldsymbol{\tau}_a\langle 1,2,3,4\rangle,\, \boldsymbol{\tau}_a\langle 0,4,5,6\rangle,\, \boldsymbol{\kappa}_{str} \lhd \boldsymbol{\tau}_a\langle 1,2,3,4\rangle \lhd \boldsymbol{\tau}_a\langle 0,4,5,6\rangle)\,]\\
&f_1(\text{path}(1,x)) = [\,(\boldsymbol{\kappa}_{str},\, \boldsymbol{\tau}_x\langle 1,2,3\rangle,\, \boldsymbol{\kappa}_{str} \lhd \boldsymbol{\tau}_x\langle 1,2,3\rangle)\,]\\
&f_1(\text{path}(2,x)) = [\,(\boldsymbol{\kappa}_{str} \lhd \boldsymbol{\tau}_a\langle 1,2,3,4\rangle,\, \boldsymbol{\tau}_x\langle 0,4,5\rangle,\, \boldsymbol{\kappa}_{str} \lhd \boldsymbol{\tau}_a\langle 1,2,3,4\rangle \lhd \boldsymbol{\tau}_x\langle 0,4,5\rangle)\,].
\end{aligned}$$

In this case, the progenitor of $\mathbf{C}_{\mathbb{N}}$ is mapped onto the progenitor of $\mathbf{C}_{str}$ corresponding to the null string, and, in general, struct $\bar{\boldsymbol{\gamma}}_n$ is mapped onto $\bar{\boldsymbol{\gamma}}_{a^{n-1}}$ (here structs are considered as paths of length 0). Furthermore, every elementary path in $\mathbf{C}_{\mathbb{N}}$ is mapped into the elementary path in $\mathbf{C}_{str}$ corresponding to the insertion of letter a at the end of the string. Finally, the termination elementary paths $\text{path}(n,x)$ are mapped onto the termination elementary paths for $\mathbf{C}_{str}$.

For each thus specified mapping f, since the path durantion is, obviously, preserved, f defines a morphism. $\Diamond$

Remark 5. We conjecture that, in general, any morphism can be uniquely and finitely specified by its values on certain "basic" paths.

Next, we check the categorical properties of morphisms, i.e. the existence of compositions of morphisms and of the identity morphism, and the associativity of the composition.

Definition 38. For two morphisms $\mathbf{f} : \mathbf{C}_1 \to \mathbf{C}_2$ and $\mathbf{g} : \mathbf{C}_2 \to \mathbf{C}_3$, the composition of $\mathbf{f}$ and $\mathbf{g}$, denoted $\mathbf{g} \circ \mathbf{f}$, is a morphism $\mathbf{h} : \mathbf{C}_1 \to \mathbf{C}_3$, where $h = g \circ f$. ▶

Lemma 29. The above definition of composition is correct, i.e. mapping h satisfies the conditions in Def. 37.

Lemma 30. The composition of morphisms is associative, i.e. for any classes $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$ and morphisms $\mathbf{f} : \mathbf{C}_1 \to \mathbf{C}_2$, $\mathbf{g} : \mathbf{C}_2 \to \mathbf{C}_3$, $\mathbf{h} : \mathbf{C}_3 \to \mathbf{C}_4$,

$$\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f}) = (\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}.$$

Lemma 31. For a class $\mathbf{C} = (T, l, \bar{\boldsymbol{\kappa}})$, the identity path mapping $\mathrm{id} : P_T \to P_T$ defines the identity class morphism $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$, i.e. for any morphisms $\mathbf{g} : \mathbf{C}_1 \to \mathbf{C}$ and $\mathbf{h} : \mathbf{C} \to \mathbf{C}_2$,

$$\mathrm{id}_{\mathbf{C}} \circ \mathbf{g} = \mathbf{g} \quad \text{and} \quad \mathbf{h} \circ \mathrm{id}_{\mathbf{C}} = \mathbf{h}.$$

Thus, the set of all classes forms a category which is natural to call **the category of classes**.

Definition 39. Two **classes** $\mathbf{C}_1$, $\mathbf{C}_2$ will be called **isomorphic**, if there exist morphisms $\mathbf{f} : \mathbf{C}_1 \to \mathbf{C}_2$, $\mathbf{g} : \mathbf{C}_2 \to \mathbf{C}_1$ such that $\mathbf{f} \circ \mathbf{g} = \mathrm{id}_{\mathbf{C}_1}$ and $\mathbf{g} \circ \mathbf{f} = \mathrm{id}_{\mathbf{C}_2}$. ▶

3.10 Subclasses and struct representations

In this (short) section, we introduce important concepts of subclass and struct representation.

Definition 40. Morphism $\mathbf{f} : \mathbf{C}_1 \to \mathbf{C}_2$ will be called a **monomorphism**, if for any pair of distinct paths p_1, p_2 that the generating process of class $\mathbf{C}_1$ can pass,

$$f(p_1) \neq f(p_2).$$

A monomorphism will be denoted by $\mathbf{f} : \mathbf{C}_1 \hookrightarrow \mathbf{C}_2$. For class $\mathbf{C}$, a **subclass** is defined as a pair $(\mathbf{f}, \mathbf{C}')$, where $\mathbf{f} : \mathbf{C}' \hookrightarrow \mathbf{C}$. ▶

Lemma 32. A morphism $\mathbf{f} : \mathbf{C}_1 \to \mathbf{C}_2$ is a monomorphism, if and only if, for any class $\mathbf{C}$ and for any morphisms $\mathbf{g}, \mathbf{h} : \mathbf{C} \to \mathbf{C}_1$, $\mathbf{g} \neq \mathbf{h}$ implies $\mathbf{f} \circ \mathbf{g} \neq \mathbf{f} \circ \mathbf{h}$.

Important remark. Suppose that an agent has knowledge of several classes, some of which are related to each other via monomorphisms. When the agent encounters a "physical" object, he attempts to construct a "representation" of the object by discovering its formative history *based* on the agent's knowledge of some classes, i.e. the classes activated by the object. The agent accomplishes this by initiating the generating process for each of these classes. Each such process is conditioned by the above relationship (via monomorphisms) between the classes and simultaneously "guided" by the agent's structural measurement devices. The function of the *structural measurement device* is to match the structs from some of the active classes against the original "physical" object. For each of the active classes, the current state, or struct, of the corresponding generating process can be called the current representation of the "physical" object with respect to that class.

Definition 41. For a struct $\bar{\gamma} \in \mathbf{C}$, the **representation of** $\bar{\gamma}$ **with respect to a subclass** $(\mathbf{f}, \mathbf{C}')$ is defined as a pair $(f^{-1}(\bar{\gamma}), \mathbf{C}')$. ▶

Note that the "explicit" representation of struct $\bar{\gamma}$ with respect to class $\mathbf{C}$ is also not unique and is associated with a particular path from the class progenitor to $\bar{\gamma}$.

4 The learning problem

In this, our "destination", section we briefly outline a tentative formulation of the learning problem.

We assume that $(\Pi, \mathcal{I})$ is a fixed inductive structure and all classes that we consider are classes in this inductive structure. Let $\mathbf{C}$ be a fixed class in $(\Pi, \mathcal{I})$ which will be called the **superclass** and from which the learning subclasses will be selected.

First, we hypothesize that there exists unique class $\mathcal{P}(\mathbf{C})$ called the **power class** of $\mathbf{C}$, which has the following properties:

- there exists a bijection between $\mathcal{P}(\mathbf{C})$ and the set all subclasses of $\mathbf{C}$; for each subclass $(\mathbf{f}, \mathbf{K})$ of $\mathbf{C}$, the corresponding element of $\mathcal{P}(\mathbf{C})$ will be denoted as $\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}$ and should be thought of as the "representation" of that subclass
- its progenitor $\bar{\boldsymbol{\kappa}}_{\mathcal{P}(\mathbf{C})}$ corresponds to class $\mathbf{C}$ itself:

$$\bar{\boldsymbol{\xi}}_{(\mathrm{id}_{\mathbf{C}},\mathbf{C})} = \bar{\boldsymbol{\kappa}}_{\mathcal{P}(\mathbf{C})}$$

- transformations of class $\mathcal{P}(\mathbf{C})$ should be thought of as the "building blocks" with which the *representation* of any subclass of $\mathbf{C}$ (including its monomorphism) can be constructed.

Thus, one can think of the generating process $G_{\mathcal{P}(\mathbf{C})}$ for this superclass as the one constructing the representations of the subclasses of $\mathbf{C}$. Note that since $\mathcal{P}(\mathbf{C})$ is uniquely defined by $\mathbf{C}$, $G_{\mathcal{P}(\mathbf{C})}$ is also uniquely defined.

Definition 42. For a subclass $(\mathbf{f}, \mathbf{K})$ of $\mathbf{C}$, a finite subset $R \subset f(K)$ will be called a **training set generated by** $(\mathbf{f}, \mathbf{K})$. Let us define the **typicality of a training set** $R = \{\bar{\boldsymbol{\alpha}}_1, \ldots, \bar{\boldsymbol{\alpha}}_n\}$ as

$$\nu_{(\mathbf{f},\mathbf{K})}(R) \stackrel{\mathrm{def}}{=} \prod_{i=1}^{n} \nu_{\mathbf{K}}(f^{-1}(\bar{\boldsymbol{\alpha}}_i)). \tag{14}$$

►

Next, let us suppose that we know the transformation system for superclass $\mathbf{C}$, which we will now consider as an "external" class, and suppose that we are observing an external generating process for the power class $\mathcal{P}(\mathbf{C})$, and, moreover, suppose that, during our observation, this process *remains in state* $\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}$. It is natural to assume that we do not know the struct $\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}$ or, in other words, we do not know the representation of subclass $(\mathbf{f}, \mathbf{K})$. Then, suppose we observed a training set $R' = \{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\}$ generated by $(\mathbf{f}, \mathbf{K})$. We consider elements of the training set R' as independent discrete random variables whose values belong to set $f(K)$,[17] and

$$\mathrm{Prob}(\bar{\gamma}_i = \bar{\gamma}) = \nu_{\mathbf{K}}(f^{-1}(\bar{\gamma})), \qquad i \in [1, n], \;\; \bar{\gamma} \in f(K). \tag{15}$$

For a finite subset $R \subset C$, let us denote by $\mathcal{E}_R$ the random event "$R' = R$", and by $\mathcal{E}_{(\mathbf{f},\mathbf{K})}$ the random event "R' is generated by subclass $(\mathbf{f}, \mathbf{K})$". Note that event $\mathcal{E}_{(\mathbf{f},\mathbf{K})}$ can also be described as "$G_{\mathcal{P}(C)}$ is in state $\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}$". By Def. 35, the probability of event $\mathcal{E}_{(\mathbf{f},\mathbf{K})}$ is

$$\mathrm{Prob}(\mathcal{E}_{(\mathbf{f},\mathbf{K})}) = \nu_{\mathcal{P}(\mathbf{C})}(\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}). \tag{16}$$

[17] See, for example, [26].

According to (14, 15), the probability of the conditional event "$R' = R$ under condition that R' is generated by subclass $(\mathbf{f}, \mathbf{K})$" is given by formula

$$\mathrm{Prob}(\mathcal{E}_R \,|\, \mathcal{E}_{(\mathbf{f},\mathbf{K})}) = \nu_{(\mathbf{f},\mathbf{K})}(R).^{18} \tag{17}$$

Now, the **learning problem** can be formulated as follows: given a finite subset $R^0 \subset C$, find subclass $(\mathbf{f}^0, \mathbf{K}^0)$ for which the conditional probability

$$\mathrm{Prob}(\mathcal{E}_{(\mathbf{f}^0,\mathbf{K}^0)} \,|\, \mathcal{E}_{R^0}) \tag{$*$}$$

is maximal. In other words, the learning problem is to find subclass $(\mathbf{f}^0, \mathbf{K}^0)$ whose generating process is most likely to generate the observed set R^0.

For conditional probability $(*)$, we have:

$$\mathrm{Prob}(\mathcal{E}_{(\mathbf{f}^0,\mathbf{K}^0)} \,|\, \mathcal{E}_{R^0}) = \frac{\mathrm{Prob}(\mathcal{E}_{R^0} \,|\, \mathcal{E}_{(\mathbf{f}^0,\mathbf{K}^0)}) \cdot \mathrm{Prob}(\mathcal{E}_{(\mathbf{f}^0,\mathbf{K}^0)})}{\mathrm{Prob}(\mathcal{E}_{R^0})}.$$

Since the training set R^0 is fixed, the denominator is also fixed, and $(*)$ is maximal whenever the numerator is maximal. From (17, 16), we obtain

$$\mathrm{Prob}(\mathcal{E}_{R^0} \,|\, \mathcal{E}_{(\mathbf{f}^0,\mathbf{K}^0)}) \cdot \mathrm{Prob}(\mathcal{E}_{(\mathbf{f}^0,\mathbf{K}^0)}) = \nu_{(\mathbf{f}^0,\mathbf{K}^0)}(R^0) \cdot \nu_{\mathcal{P}(\mathbf{C})}(\bar{\boldsymbol{\xi}}_{(\mathbf{f}^0,\mathbf{K}^0)}),$$

so the learning problem is equivalent to the following one: given R^0, find subclass $(\mathbf{f}^0, \mathbf{K}^0)$ which maximizes

$$\nu_{(\mathbf{f}^0,\mathbf{K}^0)}(R^0) \cdot \nu_{\mathcal{P}(\mathbf{C})}(\bar{\boldsymbol{\xi}}_{(\mathbf{f}^0,\mathbf{K}^0)}). \tag{$**$}$$

We will not give a precise definition of the above power class but simply note that it has to be defined as an instance of the general categorical concept of power object specified for our category of classes. Therefore, instead of evaluating the second factor in $(**)$ directly, for practical purposes, we replace this factor by a "similar" one as explained next.

Let $\bar{\boldsymbol{\tau}}$ be a transformation from the transformation set that defines class $\mathbf{K}$. Choose any itransformation $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \bar{\boldsymbol{\tau}}$, and denote by $p_{\bar{\boldsymbol{\tau}}}$ the elementary path which contains elementary ipath $(\boldsymbol{\alpha}, \boldsymbol{\tau}, \boldsymbol{\beta})$. It is easy to check that $p_{\bar{\boldsymbol{\tau}}}$ is correctly defined.

For a subclass $(\mathbf{f}, \mathbf{K})$ of $\mathbf{C}$, where $\mathbf{K} = (\{\bar{\boldsymbol{\tau}}_1, \ldots, \bar{\boldsymbol{\tau}}_m\}, l, \bar{\boldsymbol{\kappa}})$, let us call

$$\nu_{\mathbf{C}}(\mathbf{f}, \mathbf{K}) \overset{\mathrm{def}}{=} \nu_{\mathbf{C}}(f(\bar{\boldsymbol{\kappa}})) \cdot \prod_{i=1}^{m} \mu_{\mathbf{C}}(f(p_{\bar{\boldsymbol{\tau}}_i}))$$

the **typicality of subclass** $(\mathbf{f}, \mathbf{K})$, where $\mu_{\mathbf{C}}$ is the path probability measure (see Def. 36). Then, we can replace the second factor in $(**)$, which is the typicality of the representation $\bar{\boldsymbol{\xi}}_{(\mathbf{f}^0,\mathbf{K}^0)}$ of subclass $(\mathbf{f}^0, \mathbf{K}^0)$ in power class $\mathcal{P}(\mathbf{C})$, by the typicality of subclass $(\mathbf{f}^0, \mathbf{K}^0)$.

Thus, the learning problem can now be formulated as follows: given R^0, find subclass $(\mathbf{f}^0, \mathbf{K}^0)$ which maximizes

$$\nu_{(\mathbf{f}^0,\mathbf{K}^0)}(R^0) \cdot \left[\nu_{\mathbf{C}}(f(\bar{\boldsymbol{\kappa}})) \cdot \prod_{i=1}^{m} \mu_{\mathbf{C}}(f(p_{\bar{\boldsymbol{\tau}}_i})) \right].^{19}$$

[18] See Appendix for a proof of this equality.

[19] For the classical case see, for example, [27].

Preliminary analysis suggests that this optimization problem has a non-trivial solution which corresponds to a subclass $(\mathbf{f}^0, \mathbf{K}^0)$ lying "between" two extreme non-optimal solutions: one corresponding to the superclass $\mathbf{C}$, and the other one, to the smallest subclass containing R^0.

Finally, it is important to note that, as a result of the learning process, a new object representation (see Def. 41), i.e. the pair $(f^{-1}(\bar{\gamma}), \mathbf{K})$, of an object originally represented as $(\bar{\gamma}, \mathbf{C})$ may "take over": the typicality of $f^{-1}(\bar{\gamma})$ in $\mathbf{K}$ might be higher than the typicality of $\bar{\gamma}$ in $\mathbf{C}$.

5 Future work

First of all, we want to come back to the Introduction and discuss *very* briefly the questions of explanatory value of the ETS model as well as of its experimental "verification". One might say that the description of the class by means of its transformation system encapsulates the explanatory nature of the outlined model: it suggests what the representation (or "nature") of the class is, or how to "think" about a class of objects, including the interpretation of transformations as class "features". At the same time, the experimental verification of the model might also be approached from the point of view of validity of this explanatory hypothesis about the nature of class representation. It turns out that in cognitive psychology there has already accumulated some experimental evidence supporting the model (starting from the work of M.I.Posner and co-workers and J.J.Franks and J.D.Bransford [28]). We are currently working on the initial verification of this hypothesis for some classes of organic molecules. It is understood, that to properly conduct the experimental work the form of data representation must confirm to the proposed structural form of representation, which, in turn, requires rethinking of the conventional forms of representation.

As far as the theory is concerned, below we list some of the important topics (most are very substantial) for the future research.

- Investigate the axiomatics for the inductive structure.
- Develop a theory of inductive structures (potentially of great importance for mathematics and sciences).
- Introduce a natural hierarchy of (non-isomorphic) inductive structures and show that the analogues of the numeric representations are the simplest ones.
- Are there any other natural ways to introduce the transformations?
- Introduce a natural hierarchy of transformation systems.
- Is there a different/interesting approach, not relying on stochastic processes, to the generation of class elements?
- Should one allow the classes with several progenitors and how to define such classes?
- Introduce the concept of morphism for inductive structures and investigate the alternative definitions of the class morphism.
- Propose a formal definition of the power class.
- Propose an alternative formulation of the learning problem.

- Investigate various stages of the learning process (together with the relevant concepts).
- Develop a powerful machinery (and optimization theory) for efficient construction of the class transformation system during the learning process.

Acknowledgment

We would like to thank John Abela for encouraging us to undertake this work sooner rather than later.

Appendix

Here we define accurately the random variables considered in Section 4, and prove formula (17). Let $(\Omega_1, \mathcal{B}_1, \mathrm{Prob}_1)$ be a probability space, and let

$$\mathrm{sc} : \Omega_1 \to \mathcal{P}(\mathbf{C})$$

be a random variable with the values in the power class $\mathcal{P}(\mathbf{C})$ which has the following meaning: for $\omega_1 \in \Omega_1$,

$$\mathrm{sc}(\omega_1) = \bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}$$

denotes a random <u>subclass</u> $(\mathbf{f}, \mathbf{K})$ of $\mathbf{C}$ generated by the generating process of $\mathcal{P}(\mathbf{C})$.

After the generating process of $\mathcal{P}(\mathbf{C})$ generates the subclass $(\mathbf{f}, \mathbf{K})$, the generating process for subclass $(\mathbf{f}, \mathbf{K})$ generates a training set.

Let $(\Omega_2, \mathcal{B}_2, \mathrm{Prob}_2)$ be a probability space, and assume that for every subclass $(\mathbf{f}, \mathbf{K})$ of $\mathbf{C}$ there exist independent random variables

$$t_i^{\mathbf{K}} : \Omega_2 \to K, \qquad i \in [1, n]$$

corresponding to the elements of a training set for class $\mathbf{K}$.

Consider the product of the introduced probability spaces

$$(\Omega, \mathcal{B}, \mathrm{Prob}) = (\Omega_1 \times \Omega_2,\ \mathcal{B}_1 \times \mathcal{B}_2,\ \mathrm{Prob}_1 \times \mathrm{Prob}_2)$$

and random variables

$$t_i : \Omega \to C,^{20} \qquad i \in [1, n]$$

defined as

$$t_i(\omega_1, \omega_2) \stackrel{\mathrm{def}}{=} \mathbf{f}\left(t_i^{\mathbf{K}}(\omega_2)\right), \text{ where } (\mathbf{f}, \mathbf{K}) \text{ is such that } \bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})} = \mathrm{sc}(\omega_1).$$

Thus, $\{t_i(\omega_1, \omega_2) \,|\, i \in [1, n]\}$ is the image in C of a random training set for a random subclass $(\mathbf{f}, \mathbf{K})$. Suppose we observed a particular training set $R = \{\bar{\gamma}_1, \ldots, \bar{\gamma}_n\} \subset C$. The conditional probability in the left hand side of (17) can now be rewritten as

$$\begin{aligned}\mathrm{Prob}(\mathcal{E}_R \,|\, \mathcal{E}_{(\mathbf{f},\mathbf{K})}) &= \mathrm{Prob}\left[t_1(\omega_1, \omega_2) = \bar{\gamma}_1 \,,\ \ldots\ ,\ t_n(\omega_1, \omega_2) = \bar{\gamma}_n \ \middle|\ \mathrm{sc}(\omega_1) = \bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})} \right] = \\ &= \frac{\mathrm{Prob}\left[t_1(\omega_1, \omega_2) = \bar{\gamma}_1 \,,\ \ldots\ ,\ t_n(\omega_1, \omega_2) = \bar{\gamma}_n \,,\ \mathrm{sc}(\omega_1) = \bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})} \right]}{\mathrm{Prob}\left(\mathrm{sc}(\omega_1) = \bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})} \right)} = \frac{\mathrm{Prob}(X)}{\mathrm{Prob}(Y)} \,,\end{aligned}$$

[20] It is not hard to see that t_i are indeed random variables, i.e. inverse image of any element in C is a measurable subset of Ω_2.

where

$$\begin{aligned} X &= \Big\{(\omega_1,\omega_2)\in\Omega \;\Big|\; t_1(\omega_1,\omega_2)=\bar{\gamma}_1\,,\;\ldots\;,\, t_n(\omega_1,\omega_2)=\bar{\gamma}_n,\; \mathrm{sc}(\omega_1)=\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}\Big\} \\ Y &= \Big\{(\omega_1,\omega_2)\in\Omega \;\Big|\; \mathrm{sc}(\omega_1)=\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}\Big\}. \end{aligned}$$

If

$$\begin{aligned} X_1 &= \Big\{\omega_1\in\Omega_1 \;\Big|\; \mathrm{sc}(\omega_1)=\bar{\boldsymbol{\xi}}_{(\mathbf{f},\mathbf{K})}\Big\} \\ X_2 &= \Big\{\omega_2\in\Omega_2 \;\Big|\; \mathbf{f}\left(t_1^{\mathbf{K}}(\omega_2)\right)=\bar{\gamma}_1\,,\;\ldots\;,\,\mathbf{f}\left(t_n^{\mathbf{K}}(\omega_2)\right)=\bar{\gamma}_n\Big\}\,, \end{aligned}$$

then $X = X_1 \times X_2$, $Y = X_1 \times \Omega_2$, and, according to [26, Remark 35.13],

$$\begin{aligned} \mathrm{Prob}(X) &= \mathrm{Prob}_1(X_1)\cdot \mathrm{Prob}_2(X_2) \\ \mathrm{Prob}(Y) &= \mathrm{Prob}_1(X_1). \end{aligned}$$

Since $t_i^{\mathbf{K}}$, $i \in [1,n]$, are independent random variables,

$$\begin{aligned} \mathrm{Prob}_2(X_2) &= \mathrm{Prob}_2\left[\,\mathbf{f}\left(t_1^{\mathbf{K}}(\omega_2)\right)=\bar{\gamma}_1\,,\;\ldots\;,\,\mathbf{f}\left(t_n^{\mathbf{K}}(\omega_2)\right)=\bar{\gamma}_n\right] = \\ &\quad\; \mathrm{Prob}_2\left[\,\mathbf{f}\left(t_1^{\mathbf{K}}(\omega_2)\right)=\bar{\gamma}_1\right]\cdot\ldots\cdot \mathrm{Prob}_2\left[\mathbf{f}\left(t_n^{\mathbf{K}}(\omega_2)\right)=\bar{\gamma}_n\right] = \nu_{(\mathbf{f},\mathbf{K})}(R). \end{aligned}$$

Hence,

$$\mathrm{Prob}(\mathcal{E}_R \,|\, \mathcal{E}_{(\mathbf{f},\mathbf{K})}) = \frac{\mathrm{Prob}(X)}{\mathrm{Prob}(Y)} = \frac{\mathrm{Prob}_1(X_1)\,\mathrm{Prob}_2(X_2)}{\mathrm{Prob}_1(X_1)} = \nu_{(\mathbf{f},\mathbf{K})}(R),$$

and (17) is proved.

References

[1] G. Sarton, Ancient Science Through the Golden Age of Greece, Dover, New York, 1993.

[2] G. Ifrah, The Universal History of Numbers, J. Wiley, New York, 2000.

[3] E. Schrodinger, Nature and the Greeks and Science and Humanism, pp.143-145, 158, CUP, Cambridge, UK, 1996.

[4] L. Goldfarb, On the foundations of intelligent processes I: An evolving model for pattern learning, Pattern Recognition 23 (6) (1990), 595-616.

[5] L. Goldfarb, What is distance and why do we need the metric model for pattern learning, Pattern Recognition 25 (4) (1992), 431-438.

[6] L. Goldfarb, S. Nigam, The unified learning paradigm: A foundation for AI, in: V. Honavar, L. Uhr (Ed.), Artificial Intelligence and Neural Networks: Steps toward Principled Integration, 1994, pp. 533-559.

[7] L. Goldfarb, J. Abela, V. C. Bhavsar, V. N. Kamat, Can a vector space based learning model discover inductive class generalization in a symbolic environment? Pattern Recognition Letters 16 (7) (1995) 719-726.

[8] L. Goldfarb, Inductive class representation and its central role in pattern recognition, Proc. Conf. Intelligent Systems: A Semiotic Perspective, Vol. 1, NIST, Gaithersburg, Maryland, USA, 1996, pp. 53-58.

[9] L. Goldfarb, What is inductive learning?: Construction of inductive class representation, Proc. Workshop "What Is Inductive Learning" in Conjunction with 11th Biennial Canadian AI Conf., 1996, pp. 9-21.

[10] L. Goldfarb, S. Deshpande, What is a symbolic measurement process?, Proc. IEEE Conf. Systems, Man, and Cybernetics, Vol. 5, Orlando, Florida, USA, 1997, pp. 4139-4145.

[11] L. Goldfarb, J. Hook, Why classical models for pattern recognition are not pattern recognition models, Proc. Intern. Conf. On Advances in Pattern Recognition, Plymouth, UK, 1998, pp. 405-414.

[12] E. G. H. Landau, Foundations of Analysis, Chelsea, New York, 1951.

[13] C. Lee, Notes for Math 502, 1998 [http://www.ms.uky.edu/ lee/ma502/notes2/node7.html].

[14] M.A. Aiserman, Remarks on two problems connected with pattern recognition, in: S. Watanabe (Ed.), Methodologies of Pattern Recognition, , p.1, Academic Press, 1969.

[15] J. Losee, A Historical Introduction to the Philosophy of Science, 3rd ed., OUP, Oxford, 1993.

[16] A. Bird, Philosophy of Science, McGill-Queen's University Press, Montreal, 1998.

[17] J. H. Holland, K. J. Holyoak, R. E. Nisbett, P. R. Thagard, Induction, MIT Press, Cambridge, Mass., 1986.

[18] M. Piattelli -Palmarini (Ed.), Language and Learning: The Debate between Jean Piaget and Noam Chomsky, pp. 100-103, 255-272, HUP, Cambridge, USA, 1980.

[19] N. Chomsky, Knowledge of Language: Its Nature, Origin, and Use, p. 12, Praeger, New York, 1986.

[20] D. Wolpert, Preface, in: D. Wolpert (Ed.), The Mathematics of Generalization, p.8, Addison-Wesley, Reading, MA, USA, 1995.

[21] A.R. Lacey, A Dictionary of Philosophy, 3rd ed., p. 308, Routledge, London, UK, 1996.

[22] R. Dunbar, The Trouble with Science, p. 17, Faber and Faber, London, UK, 1996.

[23] N. Chomsky, Syntactic Structures, Mouton, London, The Haugue, The Netherlands, 1965.

[24] A.G. Kurosh, Lectures on General Algebra, Section 1.5, Chelsea, New York, 1963.

[25] S. MacLane, G. Birkhoff, Algebra, 3rd ed., Chelsea, New York, 1993.

[26] K.R. Parthasarathy, Introduction to Probability and Measure, Sections 3, 22, Macmillan India Press, Madras, 1977.

[27] T.M. Mitchell, Machine Learning, Sections 6.3, 6.9, McGraw-Hill, Boston, 1997.

[28] J.J. Franks, J.D. Bransford, Abstraction of visual patterns, J. of Experimental Psychology 90 (1) (1971) 65-74.