STEP BY STEP METHODS FOR THE NUMERICAL SOLUTION OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

The integro-differential equations to be considered can be expressed in the form

(1) $y'(x) = F(x,y(x),z(x))$, $y_0 = \eta$, $0 \leqq x \leqq a$

where $z(x) = \int_0^x K(x,t,y(t))dt$

Let $R_1 = \{(x,t,u) : 0 \leqq t \leqq x \leqq a \ , \ |u| < \infty \}$

and $R_2 = \{(x,y,z) : 0 \leqq x \leqq a \ , \ |y| < \infty, \ |z| < \infty \}$

If i) K and F are uniformly continuous for all $(x,t,u) \in R_1$ and $(x,y,z) \in R_2$ respectively, and

ii) the following Lipschitz conditions are satisfied

$|K(x,t,u_1) - K(x,t,u_2)| \leqq L_1 \ |u_1 - u_2|$ for all $(x,t,u_i) \in R_1, i=1,2$

$|F(x,y,z_1) - F(x,y,z_2)| \leqq L_2 \ |z_1 - z_2|$ for all $(x,y,z_i) \in R_2, i=1,2$

$|F(x,y_1,z) - F(x,y_2,z)| \leqq L_3 \ |y_1 - y_2|$ for all $(x,y_i,z) \in R_2, i=1,2$

then it is well known ([1], [6]) that equation (1) has a unique solution $y(x) \in C \ [0,a]$.

Methods for finding approximations to the solution of (1) have been considered for example in ([1], [2], [3], [4], [6]). In this report, a general class of step by step methods is considered which can be used to obtain starting values or approximations for the complete solution. Special attention will be given to one and two step exponentially fitted methods. These methods have been studied with respect to solving ordinary differential equations in papers by Nevanlinna and

Sipilä (see, for example [8]). In particular, conditions for these methods to be A-stable are presented.

2. NUMERICAL METHODS

Let I(h) denote a partition of [0,a] where

$$I(h) = \{x_i : x_i = ih , i = 0,1,\ldots,N , h > 0, Nh = a\}$$

A general one-step method for approximating the solution of (1) can be expressed in the form

$$(2) \quad y_{n+1} = y_n + hQ(x_n,y_n,z_n,h)$$

where $z_n = h \sum_{j=0}^{n-1} \Phi(x_n,x_j,y_j,h)$

Here y_n and z_n denote approximations to $y(x_n)$ and $z(x_n)$ respectively. The functions Q and Φ are called increment functions. These methods are said to be explicit if the increment function Q(x,y,z,h) requires no approximations to y(x) outside $[x_n,x_{n+1})$. Otherwise, they are said to be implicit.

To obtain a particular collection of methods, we solve the following problem taken from [8]. Find a function $u \in U$, the space spanned by $\{1,x,\ldots,x^{p-1} , e^{\gamma x}\}$, such that

$$u^{(j)}(x_n) = y_n \qquad j = 0$$

$$= F^{(j-1)}(x_n,y_n,z_n) \quad , \quad j = 1,2,\ldots p \ , \ p > 1$$

where $\{ F^{(j)}(x_n,y_n,z_n)\}$ are the known values of the derivatives of the solution of the integro-differential equation at the point $x_n \in I(h)$.

Solving this problem, we obtain

$$(3)\quad y_{n+1} = y_n + \sum_{r=1}^{p-1} \frac{h^r}{r!} F^{(r-1)}(x_n,y_n,z_n) + h^p\,\beta(\gamma h)\, F^{(p-1)}(x_n,y_n,z_n)$$

where $\beta(\gamma h) = (e^{\gamma h} - E_{p-1}(\gamma h))/(\gamma h)^p, E_{p-1}(\gamma h) = \sum_{j=0}^{p-1} (\gamma h)^j/j!$

An approximation for $z(x_n)$ can be obtained if we approximate the integral of $K(x_n,t,y(t))$ over $[x_j,x_{j+1}]$, $j = 0,1,\ldots,n-1$

by $\sum_{r=1}^{p} \frac{h^r}{r!} K^{(r-1)}(x_n,x_j,y_j)$

In section (4), z_n will be obtained using Newton-Cotes or Newton-Gregory quadrature once sufficient starting values are known.

If the function $e^{-\gamma x}$ is added to the set of generating functions for U, then we get the following formula for y_{n+1}

$$(4)\quad y_{n+1} = -y_{n-1} + 2y_n + 2\sum_{j=1}^{m} \frac{h^{2r}}{(2r)!} F^{(2r-1)}(x_n,y_n,z_n) + h^p \propto(\gamma h)\, F^{(p-1)}(x_n,y_n,z_n)$$

In (4), m is the integer part of $(p-1)/2$ and

$$\propto(\gamma h) = \beta(\gamma h) + (-1)^p \beta(-\gamma h)$$

In particular, we note that equation (4) is a two-step method. We shall see in the next section that using (3) to find y_{n+1} for even n and (4) to find y_{n+1} for odd n recovers the stability properties which hold for equation (3) but not for equation (4).

3. Convergence and Stability

Definition 1. Let $y_0, y_1, \ldots, y_N$ denote approximations to the solution of (1) by a given method with step h. Then the method is said to be convergent if $\operatorname{Max}|y_i - y(x_i)| \to 0$ as $h \to 0$, $N \to \infty$ such that $Nh = a$.

Let $$e_i = y_i - y(x_i)$$

Definition 2. A step by step method is said to be of order p if p is the largest real number for which there exists a finite positive constant C such that

$$|y_i - y(x_i)| \leq Ch^p \qquad \text{for all } x_i \in I(h)$$

Definition 3. A step by step method of the form (2) is said to be consistent with equation (1) if

$$Q(x,y,z,0) = F(x,y,z) \text{ for all } (x,y,z) \in R_2$$

Theorem 1 and its corollary are analogous to results in a text by Henrici [5]. The proofs will depend on the following three lemmas

Lemma 1. If $\{V_i\}$ is a sequence of numbers satisfying

$$|V_n| \leq A \sum_{i=0}^{n-1} |V_i| + B \qquad n = r, r+1, \ldots, N$$

with $A > 0$, $B > 0$ and $\sum_{i=0}^{r-1} |V_i| \leq V$, then

$$|V_n| \leq (B+AV)(1+A)^{n-r} \qquad n = r, r+1, \ldots, N$$

In particular, if $A = hM$, $M > 0$ and $x=nh$, then

$$|V_n| \leq (B+AV)e^{Mx}$$

Proof: See [7]

Lemma 2. Assume

a) $\Phi(x,t,y,h)$ is continuous jointly as a function of its arguments for all $(x,t,y)\varepsilon R_1$ and $h\leqq h_0$.

b) $|\Phi(x,t,y,h) - \Phi(x,t,y^*,h)| \leqq k_1 |y-y^*|$ for all (x,t,y) and $(x,t,y^*) \varepsilon R_1$ and $h\leqq h_0$.

c) $\Phi(x,t,y,0) = K(x,t,y)$ for all $(x,t,y) \varepsilon R_1$

Then

$$|z_n - z(x_n)| \leqq h \sum_{j=0}^{n-1} (k_1|e_j| + B_1 h) \text{ for all } x_n \varepsilon I(h),\ B_1>0$$

Proof. Using the mean value theorem for integrals, we have

$$(5) \quad z_n - z(x_n) = h \sum_{j=0}^{n-1} [\Phi(x_n,x_j,y_j,h) - K(x_n,x_j+\beta_j h,y(x_j+\beta_j h))],\ |\beta_j|<1$$

The expression in brackets can be written

$$\Phi(x_n,x_j,y_j,h) - \Phi(x_n,x_j,y(x_j),h)$$

$$+\Phi(x_n,x_j,y(x_j),h) - \Phi(x_n,x_j,y(x_j),0)$$

$$+K(x_n,x_j,y(x_j)) - K(x_n,x_j+\beta_j h,y(x_j+\beta_j h))$$

By assumption (a) $\Phi(x,t,y,h)$ is uniformly continuous in the compact region

$$S_1 = \{(x,t,y): \quad 0\leqq t\leqq x\leqq a,\ y = y(x)\} \text{ and } h\leqq h_0$$

Therefore, there exists a function $u(h)$ defined by

$$u(h) = 2 \max_{\substack{(x,t,y)\varepsilon S_1 \\ |t-t_1|<h\leqq h_0}} \{|\Phi(x,t,y,h) - \Phi(x,t,y,0)|,\ |K(x,t,y) - K(x,t_1,y(t_1))|\}$$

In particular for some constant $B_1>0$ and h sufficiently small

$u(h) \leq B_1 h$

The lemma follows by taking absolute values in equation (5) and using the above result.

Lemma 3. Assume

a) $Q(x,y,z,h)$ is continuous jointly as a function of its arguments $(x,y,z)\varepsilon R_2$ and $h \leq h_0$

b) $|Q(x,y,z,h) - Q(x,y^*,z,h)| \leq k_2 |y-y^*|$

$|Q(x,y,z,h) - Q(x,y,z^*,h)| \leq k_3 |z-z^*|$

for all $(x,y,z,),(x,y^*,z)$ and $(x,y,z^*) \varepsilon R_2$ and $h \leq h_0$

c) $Q(x,y,z,0) = F(x,y,z)$ for all $(x,y,z) \varepsilon R_2$

Then

$$|Q(x_n,y_n,z_n,h) - (1/h) \int_{x_n}^{x_{n+1}} F(x,y(x),z(x))\, dx|$$

$$\leq k_2|e_n| + k_3 |z_n - z(x_n)| + B_2 h,\ B_2 > 0\ ,\ \text{for all } x_n \ \varepsilon\ I(h)$$

Proof: Using the mean value theorem for integrals, we have

$$Q(x_n,y_n,z_n,h) - (1/h) \int_{x_n}^{x_{n+1}} F(x,y(x),z(x))dx$$

$$=Q(x_n,y_n,z_n,h) - Q(x_n,y(x_n),\ z_n,h)$$

$$+Q(x_n,y(x_n),z_n,h) - Q(x_n,y(x_n),\ z(x_n),\ h)$$

$$+Q(x_n,y(x_n),\ z(x_n),h)-Q(x_n,y(x_n),\ z(x_n),\ 0)$$

$$+F(x_n,y(x_n),\ z(x_n)) - F(x_n+\alpha_n h,\ y(x_n+\alpha_n h),\ z(x_n-\alpha_n h))\ ,\ |\alpha_n|<1$$

Setting up the compact region S_2 where

$$S_2 = \{(x,y,z):\quad 0 \leq x \leq a,\quad y=y(x),\quad z=z(x)\},$$

the remainder of the proof is analogous to that of lemma 2.

Theorem 1. In addition to the hypotheses of lemmas 2 and 3, assume that $|e_0| \leq \eta(h)$ where $\eta(h) \to 0$ as $h \to 0$. Then methods of the form (2) are convergent.

Proof: Integrate equation (1) over $[x_n, x_{n+1}]$ and subtract the result from equation (2). This yields

$$e_{n+1} = e_n + h \left[Q(x_n, y_n, z_n, h) - (1/h) \int_{x_n}^{x_{n+1}} F(x, y(x), z(x))dx\right]$$

Taking absolute values and using lemmas 2 and 3, this equation becomes

$$|e_{n+1}| \leq \eta(h) + hk_2 \sum_{j=0}^{n} |e_j| + hk_3 \sum_{j=0}^{n} \left(h \sum_{i=0}^{j-1} (k_1|e_i| + B_1 h)\right) + B_2 ah$$

$$\leq hM_1 \sum_{j=0}^{n} |e_j| + M_2 h + \eta(h)$$

where $M_1 = k_2 + ak_1k_3$ and $M_2 = a^2k_3B_1 + aB_2$

It follows using lemma 1 that the methods are convergent.

Corollary 1. Let p be a nonnegative integer and M be a nonnegative constant satisfying

$$\left|Q(x, y(x), z(x), h) - 1/h \int_{x}^{x+h} F(t, y(t), z(t))dt\right| \leq Mh^p$$

for each $x = x_n \in I(h)$. Assume $\eta(h) \leq \eta h^q$, for some $q, q>0$

Then methods of the form (2) have order $r = \min(p, q)$

Returning to the methods defined by equations (3) and (4), we have the following theorems

Theorem 2. Assume $F \in C^{p+1}$. Then equation (3) defines a convergent method of order p.

Proof: See [8, Theorem 1]. The proof here is analogous.

Theorem 3. Assume that $F \in C^{p+2}$ and that $|e_1| \leq Mh^{p+1}$ for some constant $M>0$. Then equation (4) is convergent and has order p for p odd and order p+1 for p even.

Proof: Consider the associated difference-differential operator $L(y(x+h),\gamma,h)$ for equation (4).

$$L\,(y(x+h),\,\gamma,\,h) = -y(x+h) + 2\sum_{r=0}^{m} (h^{2r}/(2r!))\; y^{(2r)}(x)$$
$$- y(x-h) + h^p\; (\beta(\gamma h) + (-1)^p\; \beta(-\gamma h))\; y^{(p)}(x)$$

Expanding y(x+h) and y(x-h) in a Taylor series about x and simplifying $\beta(\gamma h)$ and $\beta(-\gamma h)$ the right hand side reduces to

$$\frac{-h^p}{p!}\, y^{(p)}(x) - \frac{h^{p+1}}{(p+1)!}\, y^{(p+1)}(x) - \frac{h^{p+2}}{(p+2)!}\, y^{(p+2)}(x)$$
$$-\frac{(-h)^p}{p!}\, y^{(p)}(x) - \frac{(-h)^{p+1}}{(p+1)!}\, y^{(p+1)}(x) - \frac{(-h)^{p+2}}{(p+2)!}\, y^{(p+2)}(x)$$

$$+h^p \;\left(\frac{1}{p!} + \frac{\lambda h}{(p+1)!} + \frac{\lambda^2 h^2}{(p+2)!} \right)\; y^{(p)}(x)$$
$$+(-h)^p \;\left(\frac{1}{p!} - \frac{\lambda h}{(p+1)!} + \frac{\lambda^2 h^2}{(p+2)!} \right)\; y^{(p)}(x) + O(h^{p+3})$$

For p odd, this reduces to

$$L(y(x+h),\gamma,h) = \frac{2h^{p+1}}{(p+1)!}\; (\gamma y^{(p)}(x) - y^{(p+1)}(x)) + O(h^{p+3})$$

For p even, we have

$$L(y(x+h),\gamma,h) = \frac{2h^{p+2}}{(p+2)!}\; (\gamma^2\; y^{(p)}(x) - y^{(p+2)}(x)) + O(h^{p+3})$$

Next let us consider the problem of A-stability

Definition 4. An approximating method is said to be A-stable if when applied to the problem

$$(6) \qquad y' = \lambda + \lambda^2 \int_0^x y(t)dt \quad , \quad y_0 = 1, \ \lambda<0$$

with arbitrary step h, then

$$\lim_{\substack{n \to \infty \\ h \text{ fixed}}} y_n = 0$$

Definition 5. If an A-stable method reduces to the form $y_n = g(\lambda h)\, y_{n-1}$ when applied to equation (6) where $|g(\lambda h)| \to 0$ as $\lambda \to -\infty$, then the method is said to be strongly A-stable.

Theorem 4. A method of the form (3) with $\gamma=\lambda$ is L-stable for every $p \geq 1$.

Proof: The proof is analogous to that of theorem 4 in [8].

A similar statement for equation (4) does not hold. This is easily seen by noting that the difference equation which results when equation (5) is solved using a method of the form (4) has a root greater than one in magnitude.

Consider the method defined by the following system

$$(7a)\quad y_{n+1} = y_n + \sum_{j=1}^{p-1} \frac{h^j}{j!} F^{(j-1)}(x_n,y_n,z_n) + h^p\beta(\gamma h)\, F^{(p-1)}(x_n,y_n,z_n)$$

$$(7b)\quad y_{n+2} = -y_n + 2y_{n+1} + 2\sum_{j=1}^{m} \frac{h^{2j}}{(2j)!} F^{(2j-1)}(x_{n+1},y_{n+1},z_{n+1})$$

$$+h^p\ \alpha(\gamma h)\ F^{(p-1)}(x_n,y_n,z_n)$$

where $n = 0,2,4,\ldots.$

Theorem 5. Assume $F \in C^{p+2}$. Then the method defined by the system 7 is convergent and has order p.

Proof: By theorem 4, there exists a constant $C_1 \geq 0$ satisfying $|y_1 - y(x_1)| \leq C_1 h^{p+1}$. Also, using this result, it follows from theorem 5 that there exists a constant $C_2 \geq 0$ satisfying $|y_2 - y(x_2)| \leq C_2 h^{p+1}$. Assuming there exists a positive constant C and that $|y_i - y(x_i)| \leq Ch^{p+1}$ for $i=1,2,\ldots,k$, the proof follows easily by induction by considering the cases k even and k odd separately.

Theorem 6. The method defined by 7 with $\gamma=\lambda$ is L-stable for every $p \geq 1$.

Proof: Solving equation (6) by using the system (7) we obtain

$$y_{n+1} = e^{\lambda h}\, y_n$$

from 7a and

$$y_{n+2} = -y_n + (e^{\lambda h} + e^{-\lambda h})\, y_{n+1} = -y_n + e^{\lambda h} y_{n+1} + e^{-\lambda h}(e^{\lambda h} y_n)$$

$$= e^{\lambda h} y_{n+1}$$

from 7b. Therefore $y_k = e^{\lambda h} y_{k-1}$ for all k and the method is L-stable.

4. A NUMERICAL EXAMPLE

In this section a nonlinear example will be solved using the methods discussed and the results are compared to an A-stable method taken from [4]. To distinguish these methods, we label them as follows

Method A - the one-step method given by equation (3)

Method B - the two-step method given by system (7)

Method C - A-stable method taken from [4]

Example $y'(x) = xe^{1-y(x)} - \frac{1}{(1+x)^2} - x - \int_0^x \frac{x}{(1+t)^2} e^{1-y(t)} dt$

$y(0) = 1;$ exact solution $y(x) = 1/(1+x)$

The results are summarized in Table 1.

TABLE 1

table of absolute errors

	x	h=.05	h=.10	h=.20
Method A	1.	$3.05\text{x}10^{-6}$	$4.70\text{x}10^{-5}$	$7.06\text{x}10^{-4}$
	2.	$1.90\text{x}10^{-6}$	$2.90\text{x}10^{-5}$	$4.20\text{x}10^{-4}$
	3.	$1.84\text{x}10^{-6}$	$2.81\text{x}10^{-5}$	$4.09\text{x}10^{-4}$
	4.	$1.84\text{x}10^{-6}$	$2.81\text{x}10^{-5}$	$4.09\text{x}10^{-4}$
Method B	1.	$3.10\text{x}10^{-6}$	$4.81\text{x}10^{-5}$	$7.22\text{x}10^{-4}$
	2.	$1.92\text{x}10^{-6}$	$2.93\text{x}10^{-5}$	$4.28\text{x}10^{-4}$
	3.	$1.86\text{x}10^{-6}$	$2.84\text{x}10^{-5}$	$4.13\text{x}10^{-4}$
	4.	$1.86\text{x}10^{-6}$	$2.84\text{x}10^{-5}$	$4.13\text{x}10^{-4}$

Method C	1.	3.33×10^{-8}	4.86×10^{-7}	7.46×10^{-6}
	2.	4.87×10^{-7}	3.55×10^{-6}	2.36×10^{-5}
	3.	1.71×10^{-6}	4.31×10^{-6}	2.92×10^{-5}
	4.	1.74×10^{-6}	4.43×10^{-6}	3.00×10^{-5}

References.

1. H. Brunner, On the numerical solution of nonlinear Volterra integro-differential equations, BIT 13 (1973), 381-390

2. J.T. Day, Note on the numerical solution of integro-differential equations, Comput. J, 9 (1967), 394-395

3. Alan Feldstein and John R. Sopka, Numerical methods for nonlinear Volterra integro-differential equation, SIAM J. Numer. Anal., 11 (1974), 826-846

4. L. Garey and M. LeBlanc, Quadrature Formulae for functions of two variables and applications, Technical Report TR75-007 (1975), University of New Brunswick

5. P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, Wiley, New York, 1962

6. P. Linz, Linear multistep methods for Volterra integro-differential equations, J. Assoc. Comp. Mach, 16 (1969), 295-301

7. P. Linz, A method for the approximate solution of linear integro-differential equations, SIAM J. Numer Anal, 11 (1974), 137-144

8. O. Nevanlinna and A.H. Sipila, Explicit A-stable one-step methods for ordinary differential equations, Proc. Fourth Manitoba Conference on Numerical Math., (1974), 343-350

9. P. Pouzet, Méthode d'integration numérique des équations intégrales et intégro-differentielles du type Volterra de seconde espèce, Formules de Runge-Kutta, Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-differential Equations, Rome (1960), 362-368