McKay Quivers of the Groups $G(r, m, n)$

by

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Abstract

The following thesis proves a simple combinatorial description of the McKay quivers of the infinite family of complex reflection groups typically denoted $G(r, m, n)$. These are the groups sometimes referred to as the *finite imprimitive reflection groups*, see [4].

Our approach is combinatorial in nature - identifying the irreducible representations of the groups through the use of specific configurations of Young tableaux, and the action of the groups thereon.

This project is motivated by a desire to understand the skew group algebra: it can be shown, see Theorem 1.8 in [12], that the skew group algebra is Morita equivalent to the path algebra generated by the McKay quiver of the group. Hence, understanding the McKay quiver is a first step to understanding the path algebra and a straight-forward combinatorial description of the McKay graph is developed herein. To fully describe the skew group algebra, one must also understand the relations on the composition of arrows. This problem is not addressed here, though the author is hopeful that the techniques from this thesis could be extended to that end.

A similar description of the McKay quiver has already been developed for the classes $G(1, 1, n) = S_n$, the permutation groups on n letters, in Chapter 6 of [3].

The first three chapters of this thesis are preliminary in nature: a brief overview of the relevant parts of representation theory; a description of Young tableaux and some of their properties; and an exegesis of a paper by Ariki and Koike [1], which explicitly describes the action of the groups $G(r, 1, n)$ and their irreducible representations.
Chapter 4 describes the McKay quivers for the groups $G(r, 1, n)$ and is new work. Chapter 5 recalls some more advanced theorems from representation theory and describes the irreducible representations of $G(r, m, n)$. As such, it is not new work - though the description of the irreducibles is novel (as far as the author knows). Chapter 6 describes the McKay quivers of the groups $G(r, m, n)$.

In memory of Jilly Bean.

For Kayla: this is as much yours as mine.

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Chapter 1

Representation Theory

The following is an introduction to the very basics of representation theory. Much of what appears here has been sourced from Isaacs [8], but any sufficiently advanced text on group theory would suffice. Notation has been adjusted slightly to reconcile with [1].

It should be noted that we are concerned in this thesis with only finite groups. As such, we have the global assumption that G is a finite group.

1.1 Representations and Characters

We begin with some basic definitions and some immediate theorems.

Definition 1.1.1. Let k be a field and G a group. A $k-$representation of G is a homomorphism $\mathfrak{X} : G \longrightarrow \mathrm{GL}(n, k)$.

For our purposes, $k = \mathbb{C}$. We will often fix a basis for $\mathrm{GL}(n, \mathbb{C})$ and abuse the notation so as to refer to $\mathfrak{X}$ and its image in $\mathrm{GL}(n, \mathbb{C})$ interchangeably.

Definition 1.1.2. Let $\mathfrak{X}$ be a representation of G, then the character, χ, of G afforded by $\mathfrak{X}$ is

$$\chi(g) = \mathrm{tr}[\mathfrak{X}(g)],$$

for any $g \in G$, and where tr is the trace of the matrix $\mathfrak{X}(g)$.

Note that trace is invariant under conjugation and this assignment will thus be independent of the basis chosen. Likewise, χ will be constant on conjugacy classes of G.

Definition 1.1.3. The dimension, or degree, of a representation, $\mathfrak{X}$ will be the n in $\mathrm{GL}(\mathbb{C}, n)$. As such, it can be computed by

$$\dim\mathfrak{X} = \mathrm{tr}[\mathfrak{X}(1)] = \chi(1).$$

Definition 1.1.4. We say a character is linear if the dimension of the representation which affords it is one. That is, χ is linear if $\chi(1) = 1$.

1.2 The Group Algebra and Maschke's Theorem

In general, the group algebra is easy to define, but quite difficult to understand - at least as difficult as the group itself! Much of the business of representation theory is in representing elements of the group as linear transformations, in the hope that the tools of linear algebra can be used to understand the behaviour of the group. Roughly, the goal is to convert the rather difficult problems of group theory to the somewhat more tractable problems of linear algebra.

Note that the theorems and propositions that follow are presented without proof. These proofs can be found in [8] or your favourite text on graduate algebra.

Definition 1.2.1. Given a group G, the group algebra $\mathbb{C}[G]$, is a vector space whose basis is identified with the elements of G together with a product defined by the binary operation of G.

Fortunately, we can apply Maschke's theorem here to get a factorization of $\mathbb{C}[G]$ into irreducible submodules.

Definition 1.2.2. A module M over an algebra A is called irreducible if its only submodules are itself and the $0-$module.

Definition 1.2.3. An $A-$module, V, is called completely reducible if for any submodule $W \subset V$, there exists a complementary submodule $U \subset V$ such that $V = W \oplus U$.

Theorem 1.2.4 (Maschke)**.** *Let G be a finite group and k a field whose characteristic does not divide $|G|$. Then every $k[G]-$module is completely reducible.*

Proposition 1.2.5. *Let V be an $A-$module. Then V is completely reducible iff it is a direct sum of irreducible submodules.*

Now, since $\mathbb{C}[G]$ is then a semi-simple algebra, we have the following version of the Artin-Wedderburn theorem.

Theorem 1.2.6 (Artin-Wedderburn)**.** *The group algebra $\mathbb{C}[G]$ admits a finite list of distinct, irreducible submodules:*

$$\mathfrak{M}(\mathbb{C}[G]) = \{M_1, M_2, \cdots, M_k\},$$

such that $\mathbb{C}[G] = \bigoplus_{i=1}^{k} M_i^{\dim M_i}$.

Corollary 1.2.7. *Given $\mathbb{C}[G]$ as above, we have that*

$$\dim(\mathbb{C}[G]) = \sum_{i=1}^{k} [\dim(M_i)]^2.$$

Basically, what this is saying is that $\mathbb{C}[G] \simeq \prod_{i=1}^{k} \mathrm{End}(M_i)$; that is, $\mathbb{C}[G]$ admits a block-matrix presentation where we think of each block as an endomorphism ring of one of the irreducible M_i's.

1.3 Irreducible Representations and Characters

Fixing a basis, we can associate to each irreducible component M_i of $\mathbb{C}[G]$, a representation $\mathfrak{X}_i$.

Definition 1.3.1. We call a representation $\mathfrak{X}$ irreducible if its associated module is irreducible. We call the associated character χ an irreducible character.

This means, together with the material from the previous section, that any representation of G, $\mathfrak{X}$, can be written as a sum of the irreducible representations, $\mathfrak{X}_i$. Hence, we have that any character χ associated to $\mathfrak{X}$ can be written

$$\chi = \sum_{i=1}^{k} n_i \chi_i,$$

where the n_i are non-negative integers and the χ_i are all irreducible characters of G.

It can be shown, for instance in Theorem 2.4 in [8], that the number of conjugacy classes of G is exactly equal to the number of irreducible representations of G, which together with Corollary 1.2.7 yields the following:

Corollary 1.3.2. *The dimensions of the irreducible representations of G satisfy*

$$|G| = \sum_{i=1}^{k} \chi_i(1)^2.$$

Corollary 1.3.3. *A group G is abelian iff all of its irreducible characters are linear.*

Definition 1.3.4. Given a group G, a class function, $\varphi : G \longrightarrow \mathbb{C}$ is a mapping which is constant on the conjugacy classes of G.

Characters are themselves class functions, and any class function on G can be expressed in terms of the irreducible characters on G:

Theorem 1.3.5. *Given any class function $\varphi : G \longrightarrow \mathbb{C}$, there exists a sequence $\{a_i\}_{i=1}^k$ of complex numbers such that*

$$\varphi = \sum_{i=1}^{k} a_i \chi_i.$$

This suggests that, in some sense, the irreducible characters of G form a basis for all class functions on G. Moreoever, we will see that this basis is in fact an *orthonormal* basis.

1.4 The Character Table and Orthogonality

Definition 1.4.1. The character table associated to the finite group G is a square array whose rows are the irreducible characters of G and whose columns are paired with the conjugacy classes of G. The individual entries of the table are given by

$$a_{i,j} = \operatorname{tr}\left[\chi_i(g_j^*)\right],$$

where g_j^* is any representative from the conjugacy class $\mathfrak{K}_j$.

Example 1.4.2. Here is the character table for the group S_4, the symmetric group on four letters.

$g:$	e	(12)	$(12)(34)$	(123)	(1234)
$\lvert\mathfrak{K}\rvert:$	1	6	3	8	6
$\chi_1:$	1	1	1	1	1
$\chi_2:$	1	-1	1	1	-1
$\chi_3:$	2	0	2	-1	0
$\chi_4:$	3	1	-1	0	-1
$\chi_5:$	3	-1	-1	0	1

Technically, only the bottom right 5×5 array is the character table for S_4, but some things are added for convenience:

- the size of the conjugacy classes, $\mathfrak{K}$, will make calculations easier, as we shall see;
- the first row consists of representatives of each conjugacy class of the group: in the case of S_4, these correspond to the different partitions of 4.

Notice also that the first column of the table shows the dimension of each of the associated representations.

This table represents all of the basic ways that we can construct a map $\varphi : S_4 \longrightarrow \mathbb{C}$. For instance, χ_1 represents the map that sends every element of S_4 to the unit 1 in $\mathbb{C}$: the trivial map. χ_2 represents mapping to $\mathbb{C}$ by the sign of the permutation: we think of this as the determinant mapping.

Another way of generating a character could be by taking the permutation matrix representation of each element and simply computing its trace directly. It turns out that this is a character, but not an irreducible one for S_4: it is actually the sum of the two irreducibles χ_1 and χ_4. Because of this direct construction from the generators of the group, the representation associated to the character χ_4 is called the *standard* representation.

We claimed at the end of the last section that the irreducible characters form an orthonormal basis, but we must first have an inner product on class functions before we can substantiate such a claim.

Definition 1.4.3. Given two class functions, $\varphi,\vartheta : G \longrightarrow \mathbb{C}$, we define the pairing

$$[\varphi,\, \vartheta] = \frac{1}{|G|} \sum_{g \in G} \varphi(g)\overline{\vartheta(g)}.$$

Moreover, since these are class functions, we can simplify our calculations to

$$[\varphi,\, \vartheta] = \frac{1}{|G|} \sum_{i=1}^{k} |\mathfrak{K}_i| \varphi(g^*)\overline{\vartheta(g^*)},$$

where $g^* = g^*(\mathfrak{K})$ is a representative from each conjugacy class $\mathfrak{K}$.

It is a straight-forward exercise to show:

Proposition 1.4.4. *The pairing defined above forms an inner-product on class functions over G.*

Moreover, we have the following two propositions.

Proposition 1.4.5. *Given a list of irreducible characters* $\{\chi_1, \chi_2, \cdots, \chi_k\}$, *we have*

$$[\chi_i, \chi_j] = \delta_{i,j},$$

where $\delta_{i,j}$ *is the Kroenecker delta.*

Proposition 1.4.6. *Given two, not necessarily irreducible, characters* ψ *and* χ *we have that*

$$[\psi, \chi] = [\chi, \psi]$$

is a non-negative integer and

$$[\chi, \chi] = 1$$

iff χ *is an irreducible character.*

So we see that the irreducible characters really do form an orthonormal basis for the class functions of G and we are able to more clearly state how to express a general class function φ.

Proposition 1.4.7. *Given a finite group* G *with irreducible characters* $\{\chi_1, \chi_2, \cdots, \chi_k\}$ *we have that these form an orthonormal basis for the class functions on* G. *In fact, given a class function* $\varphi : G \longrightarrow \mathbb{C}$ *we have that*

$$\varphi = \sum_{i=1}^{k} a_i \chi_i,$$

where $a_i = [\varphi, \chi_i]$.

Remark 1.4.8. We will have occasion to refer to the tensor of two characters. What is meant is the trace of the usual outer product of the matrices corresponding to the representation evaluated on each element of G. Since, for any two matrices A and B:

$$\mathrm{tr}[A \otimes B] = \mathrm{tr}[A] \cdot \mathrm{tr}[B],$$

we have that the tensor of two representations corresponds to the component-wise product of their row-vectors in the character table.

This allows us to compute tensor products of representations and resolve them into irreducibles.

Example 1.4.9. Referring to S_4, we can use the table in Example 1.4.2 to compute $\mathfrak{X} = \mathfrak{X}_4 \otimes \mathfrak{X}_4$ and resolve it into irreducibles. This tensor translates to a component-wise product of the row vectors from the character table of S_4.

We compute:

$$\begin{aligned}\chi = \chi_4 \otimes \chi_4 &= \langle 3, 1, -1, 0, -1 \rangle \otimes \langle 3, 1, -1, 0, -1 \rangle \\ &= \langle 9, 1, 1, 0, 1 \rangle .\end{aligned}$$

Next we compute the norm of χ:

$$\begin{aligned}|\chi| = [\chi,\,\chi] &= \frac{1}{|G|}\sum_{i=1}^{k} |\mathfrak{K}_i|\ \chi(g^*)\overline{\chi(g^*)} \\ &= \frac{1}{24}\cdot(1\cdot 9\cdot 9+6\cdot 1\cdot 1+3\cdot 1\cdot 1+8\cdot 0\cdot 0+6\cdot 1\cdot 1) \\ &= 4.\end{aligned}$$

As in the breakdown of the group itself into its irreducibles, this 4 means two possible scenarios are the case. Either $\mathfrak{X}$ is made up of 4 distinct irreducible representations, or $\mathfrak{X}$ consists of a single irreducible representation repeated twice.

So, now we compute the pairing of χ with χ_3:

$$\begin{aligned}[\chi_3,\chi] &= \frac{1}{|G|}\sum_{i=1}^{k} |\mathfrak{K}_i|\ \chi_3(g^*)\overline{\chi(g^*)} \\ &= \frac{1}{24}\cdot(1\cdot 2\cdot 9+6\cdot 1\cdot 0+3\cdot 1\cdot 2+6\cdot 0\cdot(-1)+8\cdot 0\cdot 1) \\ &= 1.\end{aligned}$$

This tells us that $\mathfrak{X}_3$ appears only once in the decomposition of $\mathfrak{X}$. Further computation yields:

$$\begin{aligned}[\chi_1,\,\chi] &= 1, & [\chi_2,\,\chi] &= 0, \\ [\chi_4,\,\chi] &= 1, & [\chi_5,\,\chi] &= 1,\end{aligned}$$

so then, we have that $\mathfrak{X} = \mathfrak{X}_4 \otimes \mathfrak{X}_4 = \mathfrak{X}_1 \oplus \mathfrak{X}_3 \oplus \mathfrak{X}_4 \oplus \mathfrak{X}_5$. We will often abuse notation and write:

$$\chi_4 \otimes \chi_4 = \chi_1 \oplus \chi_3 \oplus \chi_4 \oplus \chi_5.$$

1.5 Frobenius Reciprocity

A natural question to ask at this point is: given a subgroup H of G, how are the irreducible representations of H related to those of G?

Definition 1.5.1. Given $H < G$ and χ, a character on G, we can define the *restriction*, χ_H, of χ to H, by simply evaluating χ on the elements that appear in H. This will clearly yield a class function on H which will therefore have a decomposition

$$\chi_H = \sum_{\psi} n_\psi \psi,$$

where the sum is defined over all irreducible representations of H, ψ, and

$$n_\psi = [\chi, \psi]_H,$$

where the subscript indicates that the pairing is computed in H, rather than G.

Example 1.5.2. Let us compute the character table for $S_3 < S_4$. As we already have the character table for S_4, we are able to simply restrict our conjugacy classes to those that appear in S_3:

$g:$	e	(12)	(123)
$\lvert\mathfrak{K}\rvert:$	1	3	2
$(\chi_1)_H:$	1	1	1
$(\chi_2)_H:$	1	-1	1
$(\chi_3)_H:$	2	0	-1
$(\chi_4)_H:$	3	1	0
$(\chi_5)_H:$	3	-1	0

.

It is easy to see that the first three representations remain irreducible after restriction to S_3, whereas $(\chi_4)_H$ and $(\chi_5)_H$ both split into $\chi_1 \oplus \chi_3$ and $\chi_2 \oplus \chi_3$, respectively, so that the character table of S_3 is:

$g:$	e	(12)	(123)
$\lvert\mathfrak{K}\rvert:$	1	3	2
$\chi_1:$	1	1	1
$\chi_2:$	1	-1	1
$\chi_3:$	2	0	-1

.

Remark 1.5.3. Notice that this means the dimension of a representation will be unchanged by restriction: that is, given a character χ of G and a subgroup $H < G$,

$$\chi(1) = \chi_H(1).$$

Definition 1.5.4. Given $H < G$ and χ, a character on H, we can define a character on G, called the *induced* character, χ^G:

$$\chi^G(g) = \frac{1}{|H|} \cdot \sum_{x \in G} \chi^0(xgx^{-1}),$$

where

$$\chi^0(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}.$$

Remark 1.5.5. Notice that this implies that, for a character χ on H, $\chi^G(1) = |G : H| \cdot \chi(1)$. That is, the dimension of an induced representation will be the dimension of the representation of H multiplied by the index of H in G.

The following is a fundamental theorem in character theory:

Theorem 1.5.6 (Frobenius Reciprocity)**.** *Let $H < G$, suppose that φ is a class function on H and that ϑ is a class function on G. Then,*

$$[\varphi, \vartheta_H]_H = [\varphi^G, \vartheta]_G.$$

Note that the pairing on the left is computed over H and the pairing on the right is computed over G.

It is sometimes expedient to refer to restriction and induction on the level of the representations themselves.

Definition 1.5.7. Given a group G, a subgroup $H < G$ and a representation $\mathfrak{X}$ of G. The restriction of $\mathfrak{X}$ to H is computed simply by evaluating $\mathfrak{X}$ only on elements of H. That is

$$\mathfrak{X}|_H = \mathfrak{X}(h), \text{ where } h \in H.$$

The induced representation of $\mathfrak{X}$ is then defined implicitly as the direct sum of all of the representations that restrict to $\mathfrak{X}$.

Notation 1.5.8. Given a representation, $\mathfrak{X}$ of a group G and $\mathfrak{Y}$ a representation of $H < G$:

- the restriction of $\mathfrak{X}$ to H will be denoted $\mathrm{Res}_H^G \mathfrak{X}$,
- and the representation of G induced from $\mathfrak{Y}$ will be denoted $\mathrm{Ind}_H^G \mathfrak{Y}$.

We can then recast the statement of the above Theorem 1.5.6 on the level of representations by:

$$\mathrm{Ind}_H^G \left[\mathrm{Res}_H^G(\mathfrak{X}) \otimes \mathfrak{Y} \right] \simeq \mathfrak{X} \otimes \mathrm{Ind}_H^G(\mathfrak{Y}).$$

1.6 McKay Quivers

The goal of this thesis is to provide an efficient combinatorial description of the McKay quiver of the complex reflection groups $G(r, m, n)$. These groups will be more thoroughly described later in this thesis.

Definition 1.6.1 (McKay Quiver)**.** Given a finite group, G, with irreducible representations, $\{\mathfrak{Y}_i\}_{i=1}^k$, and any representation of G, $\mathfrak{X}$, we define the McKay quiver of G with respect to the representation $\mathfrak{X}$, $\mathrm{McK}(G, \mathfrak{X})$, as follows:

- $\mathrm{McK}(G, \mathfrak{X})$ is a directed graph;
- $\mathrm{McK}(G, \mathfrak{X})$ has exactly one node for each irreducible representation of G;
- there will be an arrow from $\mathfrak{Y}_i$ to $\mathfrak{Y}_j$ for every time that $\mathfrak{Y}_j$ appears in the decomposition of the representation $\mathfrak{Y}_i \otimes \mathfrak{X}$.

Remark 1.6.2. There are a few things to note here:

- Computing the McKay quiver thus requires resolving tensor products into their irreducible components. As noted above, this is a relatively simple, if somewhat time-consuming, exercise in linear algebra.
- Note also that we will often refer to *the* McKay quiver of G, suppressing the $\mathfrak{X}$ from the definition. What is meant by this is the quiver generated by the *standard* representation of G.
- We will be computing the arrows of the quiver by pairings of the form $[\chi \otimes \sigma, \varphi]$, where σ is the character of the standard representation: in the language of quivers we will call the representation on the left of the pairing (χ) the *source* and the representation on the right (φ) the *target*.

- It should be pointed out that the standard representation does not, in general, have a rigorous definition: for a group which is linear, however, (that is, a group with a natural embedding into $\mathrm{GL}(n, \mathbb{C})$) what is typically meant by the standard representation is the representation $\mathrm{id} : G \longrightarrow \mathrm{GL}(n, \mathbb{C})$

Chapter 2

Young Diagrams

In this chapter we will recall some of the basic properties of Young diagrams and illuminate their connection to the representations of the complex reflection groups $S_n = G(1, 1, n)$, the permutation groups.

What appears here merely scratches the surface of the applications of Young diagrams - see Fulton [5] for a more in-depth treatment of Young tableaux.

2.1 Young Diagrams

Definition 2.1.1. A partition, Λ, of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n. That is, $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$, such that $\lambda_i \geq \lambda_{i+1}$, $\forall i$ and $\sum_i \lambda_i = n$.

We will often write $\Lambda \vdash n$ to mean Λ is a partition of n.

The partitions of n are of interest because they naturally correspond to the conjugacy classes of S_n.

Example 2.1.2. It is well-known that the conjugacy classes of S_n can be enumerated by the different cycle-types that appear. For instance the permutation

$$\sigma = (13)(245)$$

from S_5 corresponds to the partition

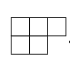
$$\Lambda = \{3, 2\}.$$

A partition can then be represented visually by a Young diagram.

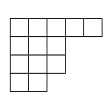
Definition 2.1.3. Given a partition of n, Λ, the Young diagram associated to Λ is defined to be a collection of cells arranged in left-justified rows whose lengths are the elements of the partition Λ.

This is more easily seen by an example.

Example 2.1.4. The Young diagram corresponding to the conjugacy class in Example 2.1.2 is

.

Example 2.1.5. For a larger example, let $\Lambda = (5, 3, 3, 2)$ be a partition of 13, then the corresponding Young diagram is



.

These partitions uniquely define the Young diagram so, where no confusion is possible, we will slightly abuse notation and simply refer to Λ as the Young diagram or partition interchangeably.

2.2 Young Tableaux

In this section, we define a Young tableau and a standard Young tableau.

Definition 2.2.1. A Young *tableau* is a Young diagram, of a given size n, that has its cells populated with numbers between 1 and n, subject to the following constraints:

- numbers that appear in the same row, must be *weakly* increasing and;
- numbers in the same column must be *strictly* increasing.

Definition 2.2.2. A *standard* Young tableau is a Young tableau that uses the numbers between 1 and n exactly once each.

Example 2.2.3. For instance, the Young diagram $\ydiagram{2,2}$ of size 4 has only two possible standard tableaux: $\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$ and $\begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau}$, while there are *many* possible tableaux, for instance:

$$\begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau}, \begin{ytableau} 1 & 2 \\ 2 & 3 \end{ytableau}, \text{ and } \begin{ytableau} 2 & 2 \\ 3 & 3 \end{ytableau},$$

are possible tableaux for this diagram, among others!

It is a well-known result that the irreducible representations of the permutation group on n objects, S_n, are not just enumerated by but, in fact, are explicitly associated to each possible Young diagram of size n. Moreover, the dimension of each irreducible representation is given by the number of possible standard tableaux on its associated Young diagram. A formula computing the dimension is given in the following section, see Theorem 2.3.2.
In fact, Young tableaux can be used to calculate *every* entry of the character table of S_n, not just the dimension (see Corollary 7.3.4 in Fulton's [5]), but understanding dimension is enough for our purposes.

The natural question then becomes: how many possible standard tableaux are there for a given Young diagram?

2.3 The Hook Length Formula

Given a Young diagram, Λ, we will refer to individual cells in the diagram by the pair (i, j) where i represents the row of the cell and j the column.

Definition 2.3.1. Given a cell (i, j) in the diagram Λ, the *hook*, $H(i, j)$, is the cell (i, j) together with all of the cells to the right and below (i, j) in the diagram.
The *hook length*, $h(i, j)$, is the number of cells in $H(i, j)$.

Theorem 2.3.2. *Let V_Λ be an irreducible representation of S_n represented* *by the Young diagram* Λ*. Then*

$$\dim V_\Lambda = \frac{n!}{\prod_{i,j} h(i,j)},$$

where (i,j) *runs over each cell in the diagram* Λ*.*

Example 2.3.3. If we wanted to know the dimension of the irreducible representation of S_6 associated to the diagram $\Lambda = \ydiagram{4,2}$, we can populate the cells with their hooklengths, $\young(5421,21)$, (note that this is *not* a tableau!) and easily compute:

$$\dim V_\Lambda = \frac{6!}{5 \cdot 4 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 9.$$

This formula is quite remarkable for its simplicity, and yet no elementary proof is known. It first appeared in a paper by Frame, Robinson and Thrall from 1954, but their proof relies on techniques from representation theory and the hooks appear in a rather non-obvious way.

A short, probabilistic, proof from Greene, Nijenhuis and Wilf, [6], is fairly easy to follow and is inspired by the following heuristic argument: if we were placing the n distinct digits randomly in the diagram, then there would be $n!$ possible configurations. However, given our restrictions on tableaux, the probability of the smallest digit in a given hook being correctly placed (that is, in the top left corner) is one in $h(i,j)$. This heuristic is quite insufficient - it would require that the placement be independent, for instance - but the proof explicitly handles the conditional probablities to prove the formula.

Chapter 3

Representations of $G(r, 1, n)$

In this chapter we will elucidate the results of a paper by Ariki and Koike, [1], and describe the irreducible representations of $G(r, 1, n)$ as well as identify the standard representation, V_{std}.

3.1 The Groups $G(r, 1, n)$

In the interest of making our discussion somewhat concrete, let us discuss the group $G(r, 1, n)$.

The permutation group of order n are concretely understood to be the permutation matrices of size $n \times n$. These matrices have the property that each row and column has a single 1 and the rest of the matrix is populated with zeroes. The groups $G(r, 1, n)$ are a generalisation of this group which admit a similar matrix presentation but allow for r^{th} roots of unity in the place of merely ones. This yields a useful matrix factorization: letting A be an element of $G(r, 1, n)$ in matrix form, then there is an $n \times n$ permutation matrix, P, and an $n \times n$ diagonal matrix whose entries are r^{th} roots of unity, D, so that $A = PD$.

Let $B = SE$ be another element in $G(r, 1, n)$, then the wreath product is computed:

$$A \cdot B = (PD) \cdot (SE) = (P(S \cdot S^{-1})D) \cdot (SE) = (PS)[(S^{-1}DS) \cdot E].$$

Which is, again, of the appropriate form.
This makes clear the typical abstract characterisation of these groups by $G(r, 1, n) = S_n \ltimes (\mu_r)^n$, the wreath product of S_n over $(\mu_r)^n$: that is, the elements of $G(r, 1, n)$ act by permuting $n-$tuples of r^{th} roots of unity.

Now, in [1], Ariki and Koike describe the generators and relations of $G(r, 1, n)$ in a way that is meant to evoke the Coxeter presentation of S_n:

Proposition 3.1.1. *The groups $G(r, 1, n)$ are characterized by the following generators and relations:*

Generators: $t = s_1, s_2, \cdots, s_n$

Relations:

$$t^r = 1$$

$$s_2^2 = s_3^2 = \cdots = s_n^2 = 1$$

$$t \cdot s_2 \cdot t \cdot s_2 = s_2 \cdot t \cdot s_2 \cdot t$$

$$s_i \cdot s_j = s_j \cdot s_i \qquad \text{if } |i-j| \geq 2$$

$$s_i \cdot s_{i+1} \cdot s_i = s_{i+1} \cdot s_i \cdot s_{i+1} \quad \text{if } i = 2, 3, \ldots n-1$$

Remark 3.1.2. Fixing ξ, a primitive $r^{th}-$root of unity, and fixing an ordered basis, we can understand the generators of the group by their matrix presentation

$$t = \text{diag}\{\xi, 1, \cdots, 1\},$$

an $n \times n$ diagonal matrix, and the other s_i, for $i > 1$

$$s_i = \text{diag}\{1, \cdots, 1, P, 1, \cdots, 1\},$$

a block-diagonal matrix with $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the $(i-1)^{th}$ position. That is, s_i is the permutation matrix which exchanges the basis vectors in positions $i-1$ and i.

Example 3.1.3. A familiar example of this type of group are the Coxeter groups $B_n = G(2, 1, n)$, which can be thought of as the symmetries of the $n-$cube. Note that these groups are somewhat unique in this context (that is, when $m = 1$) in that they are the only groups (other than S_n) on the list that are true reflection groups. Relaxing the condition on m can yield others: see [4].

Arike and Koike go on to demonstrate (see the discussion after Proposition 2.1 in [1]) a similar result to that explained in the previous chapter for S_n; the irreducible representations of $G(r,1,n)$ correspond to $r-$tuples of Young diagrams, in the following way:

Proposition 3.1.4. *To a given $r-$tuple of Young diagrams, α, we associate the space V_α which is the formal linear combination of all possible standard Young tableaux of shape α, together with a $G-$action explicitly described below.*
These V_α's are exactly the irreducible representations of $G(r,1,n)$.

Remark 3.1.5. Some clarification may be in order: the main conceit of Ariki and Koike's argument is that the vector spaces V_α are spanned *by* the $r-$tuples of standard Young tableaux of shape α. This way, the basis vectors themselves encode the action of the group in the combinatorics of the $r-$tuples.

Notation 3.1.6. Throughout, we will be using the following:

- we think of α as a 'shape' parameter: a fixed $r-$tuple of Young diagrams;
- for a given $r-$tuple, some entries will sometimes have no cell, an empty Young diagram, that we will denote with a dash: '$-$';

- we denote a particular standard Young tableau by a bold $\mathbf{t}$. This is used to reinforce that the $\mathbf{t}$'s arethe basis *vectors* of the space V_α.

Example 3.1.7. If we let $\alpha = \left(\ydiagram{2,1}, \ydiagram{1}, -\right)$ be a triple of Young diagrams of size 4. Then V_α, the corresponding irreducible representation of $G(3,1,4)$, would be the span of vectors:

$$\left\{ \left(\ytableaushort{12,3}, \ytableaushort{4}, -\right), \left(\ytableaushort{13,2}, \ytableaushort{4}, -\right), \left(\ytableaushort{12,4}, \ytableaushort{3}, -\right), \left(\ytableaushort{14,2}, \ytableaushort{3}, -\right), \right.$$
$$\left. \left(\ytableaushort{13,4}, \ytableaushort{2}, -\right), \left(\ytableaushort{14,3}, \ytableaushort{2}, -\right), \left(\ytableaushort{23,4}, \ytableaushort{1}, -\right), \left(\ytableaushort{24,3}, \ytableaushort{1}, -\right) \right\},$$

demonstrating that V_α is 8−dimensional.

More generally, it is a trivial exercise to adapt the proof from [6] to show that the hook length formula applies here as well.

Theorem 3.1.8. *Let V_α be an irreducible representation of $G(r,1,n)$ represented by the r−tuple of Young diagrams α. Then,*

$$\dim V_\alpha = \frac{n!}{\prod_{i,j} h(i,j)}.$$

Where (i,j) runs over every cell in the r−tuple α.

Example 3.1.9. If we populate the cells of the triple from the previous example with their hook lengths, $\left(\ytableaushort{31,1}, \ytableaushort{1}, -\right)$, we have

$$\dim V_\alpha = \frac{4!}{3 \cdot 1 \cdot 1 \cdot 1} = 8,$$

which agrees with our calculation.

3.2 The Action of The Generators

In [1], the action of the generators s_k on the abstract vectors, $\mathbf{t}$, defined above is explicitly demonstrated. Remember that the vectors are represented by $r-$tuples of standard Young tableaux.

Let $\mathbf{t}_p$ be one such tableau and $\mathbf{t}_q$ be the same tableau on the same diagram, but with the digits k and $k-1$ interchanged (assuming that this interchange yields a legal Young tableau). If we fix ξ, a primitive $r^{th}-$root of unity, then

$$s_1\mathbf{t}_p = \xi^{i-1}\mathbf{t}_p,$$

where the digit '1' appears in the i^{th} position of the $r-$tuple. For $k > 1$ we have the cases:

$$s_k\mathbf{t}_p = \begin{cases} \mathbf{t}_p & \text{if } k \text{ and } k-1 \text{ are in the same row,} \\ -\mathbf{t}_p & \text{if } k \text{ and } k-1 \text{ are in the same column,} \\ \mathbf{t}_q & \text{otherwise.} \end{cases}$$

The paper also demonstrates the action of the elements t_k, which are contructed iteratively by

$$t_1 = s_1 = t$$

$$t_k = s_k \cdot t_{k-1} \cdot s_k \qquad 1 < k \leq n$$

These elements act on the vectors $(\mathbf{t}_p)$ by

$$t_k \mathbf{t}_p = \xi^{i-1} \mathbf{t}_p,$$

where the digit k appears in the i^{th} position of the $r-$tuple.

Remark 3.2.1. This, together with the diagonal matrix presentation of the t_k's shows that the vectors $\mathbf{t}_p$ are the eigenvectors of the abelian subgroup $(\mu_r)^n < G$.

3.3 The Trivial and Standard Representations

The goal of this thesis is to generate the McKay graph of $G(r, 1, n)$ using the standard representation, V_{std}. The *standard* representation is taken to mean, from the perspective of the matrix presentation of $G(r, 1, n)$, that representation for which the action of the generators produce exactly the generating matrices of G. To that end, we must figure out *which* representations are playing the role of the trivial and standard representations.

Let V_τ be the space spanned by formal linear combinations of standard Young tableaux on the following Young diagram

$$\tau = (\square\square\cdots\square, -, \cdots, -) .$$

Proposition 3.3.1. *V_τ as defined above is the trivial representation of $G(r, 1, n)$.*

Proof. It is clear that V_τ is one-dimensional: there is only one possible standard tableau on τ

$$\mathbf{t} = (\boxed{1}\boxed{2}\cdots\boxed{n}, -, \cdots, -) .$$

So now we need to know how the generators act on this space:

$$s_1 = t_1 : \ \mathbf{t} \longmapsto 1 \cdot \mathbf{t},$$

since the digit '1' appears in the first component of the $r-$tuple, and

$$s_k : \ \mathbf{t} \longmapsto \mathbf{t}, \text{ for all } k > 1,$$

since every digit appears in the same row as every other digit.Thus we see that $\mathbf{t}$ is fixed by the action of the generators.

Hence, V_τ is the trivial representation of $G(r, 1, n)$, as claimed. □

Let V_α be the space spanned by the abstract vectors with the Young diagram

$$\alpha = (\square\square\cdots\square, \square, -, \cdots, -) .$$

Let the vector $\mathbf{t}_i$ represent the tableau with the numeral i appearing in the single cell in the second position of the $r-$tuple.That is

$$\mathbf{t}_i = (\square\square\cdots\square, \boxed{i}, -, \cdots, -)\,.$$

Given the constraints of Young tableaux, the other $n-1$ cells are uniquely determined by that choice. Then we have that $V_\alpha = \operatorname{span}\{\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_n\}$.

Proposition 3.3.2. *The irreducible representation V_α as defined above is the standard representation of $G(r,1,n)$.*

Proof. It is obvious that $\dim V_\alpha = n$.

From here, we can explicitly construct the matrices associated to the generators of $G(r,1,n)$. For s_1 we have that:

$$\begin{array}{rccc} s_1 = t_1 : & \mathbf{t}_1 & \longmapsto & \xi \cdot \mathbf{t}_1 \\ & \mathbf{t}_2 & \longmapsto & 1 \cdot \mathbf{t}_2 \\ & \vdots & \vdots & \vdots \\ & \mathbf{t}_n & \longmapsto & 1 \cdot \mathbf{t}_n \end{array}$$

So the matrix associated to s_1 in this basis of V_α is the diagonal matrix with ξ in the first position and 1's elsewhere.

For the other generators s_k, with $k > 1$:

$$\begin{aligned} s_k: \quad \mathbf{t}_k &\longmapsto \mathbf{t}_{k-1} \\ \mathbf{t}_{k-1} &\longmapsto \mathbf{t}_k \\ \mathbf{t}_i &\longmapsto \mathbf{t}_i \quad \text{otherwise} \end{aligned}$$

Note that one of the possibilities for the action of s_k never happens for this particular $r-$tuple, since *all* of the columns have only one cell. Thus, the matrix associated to s_k on this basis of V_α is the permutation matrix that interchanges positions k and $k-1$.

We can clearly see that these matrices are exactly the generators for the matrix presentation of $G(r,1,n)$, which proves that $V_\alpha = V_{std}$, as claimed. □

Chapter 4

The McKay Quiver of $G(r, 1, n)$

Our goal is to describe the McKay quiver of the complex reflection groups $G(r, 1, n)$ in a concise way. The irreducible representations of these groups have been described in Proposition 3.1.4, but computing tensor products in these spaces tends to be a time-consuming exercise in linear algebra. The following will prove a very simple algorithm for computing tensors with the standard representation and thus the McKay quiver. Note that the author refers to *the* McKay graph of G meaning, specifically, the one generated by tensor products with the standard representation of G, which will be referred to herein as V_{std}.

4.1 Restriction-Induction

The computation of tensors of representations is often achieved by an appeal to induction and restriction of characters of subgroups of a given group. To this end, we introduce the following notation and proposition:

Notation 4.1.1.

- We say that a cell in a Young diagram is 'available' if that cell could be deleted resulting in a legal Young diagram. Note that, from the perspective of tableaux, this means any cell that *could* be labeled with an n in a *standard* tableau on α.

- Given two $r-$tuples of Young diagrams α and β, the notation $\beta \subset \alpha$ indicates that the $r-$tuple β can be obtained from α by merely deleting cells from the Young diagrams in α.

- $\alpha \setminus \beta$ indicates the $r-$tuple of (skew) Young diagram resulting from the removal of all of the cells in β from α. Note that this operation can only be defined if $\beta \subset \alpha$.

Proposition 4.1.2. The Branching Rule: *Let α be an irreducible representation of $G(r,1,n)$. The restriction functor from representations of $G(r,1,n)$ to representations of $G(r,1,n-1)$ is given by*

$$\text{Res } V_\alpha = \bigoplus_{\substack{\beta \subset \alpha \\ |\alpha \setminus \beta| = 1}} V_\beta,$$

where β runs over all possible $r-$tuples obtained by deleting a single available cell of α.

This appears in [1] as Corollary 3.12, though we have modified it slightly to agree with previous notation in this thesis.

Example 4.1.3. Consider the representation corresponding to the $r-$tuple

$$\alpha = \left(\ydiagram{2,1}, \square, -\right)$$

from $G = G(3, 1, 4)$. Let $H = G(3, 1, 3)$, then we have that

$$\alpha_H = \left(\ydiagram{1,1}, \square, -\right) \oplus \left(\ydiagram{2}, \square, -\right) \oplus \left(\ydiagram{2,1}, -, -\right).$$

Restriction-induction in this way is close to what we need, but not quite. This is because restriction does not change the dimension of a representation, while induction increases the dimension of a representation by a factor equal to the index of the subgroup. It is easily seen that this induction will multiply the dimension by $n \cdot r$, while tensoring with V_{std} should only increase the dimension by n (the dimension of V_{std}). So what we need is a *larger* subgroup to restrict to.

Proposition 4.1.4. *Consider the subgroup $G(r, 1, n-1) \times \mu_r \hookrightarrow G(r, 1, n)$, where $\mu_r =< \xi >$ is the cyclic multiplicative group generated by ξ, a fixed,* *primitive r-th root of unity.*

The irreducible representations of $G(r, 1, n-1) \times \mu_r$ are of the form $\alpha \boxtimes i$ where α is an $r-$tuple of Young diagrams and i is a positive integer between 0 and $r-1$.

This notation will be explained in the proof.

Proof. We think of the elements of $G(r, 1, n-1) \times \mu_r$ as the formal product (PD, i), where P is an $(n-1) \times (n-1)$ permutation matrix and D an $(n-1) \times (n-1)$ diagonal matrix whose entries are powers of ξ. The embedding $G(r, 1, n-1) \times \mu_r \hookrightarrow G(r, 1, n)$, then, is then given by

$$(PD, \xi^i) \longmapsto \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} D & 0 \\ 0 & \xi^i \end{bmatrix}.$$

Now, to see the form of the irreducible representations of $G(r, 1, n-1) \times \mu_r$ we need only look at the character table of $G(r, 1, n-1) \times \mu_r$.

Letting $[G]$ be the character table of $G(r, 1, n-1)$ it is clear that the character table of $G(r, 1, n-1) \times \mu_r$ should be the Kroenecker product of $[G]$ and the character table of μ_r (see Theorem 19.18, [9]). Now, μ_r is a cyclic group of order r, so all of its representations are one-dimensional and we can explicitly write the character table of $G(r,1,n-1)\times\mu_r$:

$$\begin{bmatrix} [G] & [G] & [G] & \cdots & [G] \\ [G] & \xi[G] & \xi^2[G] & \cdots & \xi^{r-1}[G] \\ [G] & \xi^2[G] & \xi^4[G] & \cdots & \xi^{2(r-1)}[G] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [G] & \xi^{r-1}[G] & \xi^{2(r-1)}[G] & \cdots & \xi^{(r-1)(r-1)}[G] \end{bmatrix}.$$

After reducing the powers of ξ modulo r, it is clear that an irreducible representation of $G(r,1,n-1)\times\mu_r$ will consist of an irreducible representation of $G(r,1,n-1)$ together with a power ξ^i where $0\le i\le r-1$. We denote these pairs by $\alpha\boxtimes i$.

We think of the power of ξ as referring to the 'block-row' of the above character table (*i.e.* the 'row' generated by that power of ξ). □

Now we need to know how to resolve tensors of these powers.

Lemma 4.1.5. *Given two irreducible representations, $\alpha\boxtimes i$ and $\beta\boxtimes j$ of $G(r,1,n-1)\times\mu_r$ we resolve tensor products component-wise. That is,*

$$(\alpha\boxtimes i)\otimes(\beta\boxtimes j)=(\alpha\otimes\beta)\boxtimes(i\otimes j)=(\alpha\otimes\beta)\boxtimes k,$$

where $k=i+j \mod r$.

Proof. This is clear by appeal to the character table of $G(r,1,n-1)\times\mu_r$.

We think of the new power k of ξ as selecting the new block-row of the table, then $\alpha \otimes \beta$ is resolved normally within that 'row'. $\square$

Proposition 4.1.6. *The restriction functor* $\mathrm{Res}^{G(r,1,n)}_{G(r,1,n-1)\times\mu_r}$ *is given by*

$$\alpha \longmapsto \sum_{\substack{\beta\subset\alpha \\ |\alpha\setminus\beta|=1}} \beta \boxtimes (i(\alpha\setminus\beta)-1),$$

where β runs over all possible ways of (legally) deleting a single cell from α and $i(\alpha\setminus\beta)$ returns the position in the $r-$tuple of the single cell in $\alpha\setminus\beta$.

Proof. The restriction β from α is clear, given the branching rule for $G(r,1,n)$, so we need only understand the integer $i(\alpha\setminus\beta)-1$.

The key is to recall the vectors $\mathbf{t}$, the Young tableaux of shape α which span V_α. When we are deleting a cell from α we are explicitly deleting the cell from $\mathbf{t}$ which is labelled with n. Recall the action t_n from the presentation of $G(r,1,n)$ in Section 3.2,

$$t_n \cdot \mathbf{t} \longmapsto \xi^{i-1}\mathbf{t},$$

where n appears in the i^{th} component of $\mathbf{t}$.

This demonstrates our correspondence and proves the proposition. $\square$

What we have shown is that the restriction to this subgroup consists of the deletion of an available cell together with an integer which serves to 'remember' from whence we deleted the cell.

Example 4.1.7. Given the representation $\alpha = \left(\yng(2,1), \yng(2,1,1), \yng(1)\right)$ in $G(3,1,8)$ we have

$$
\begin{aligned}
\mathrm{Res}^{G(3,1,8)}_{G(3,1,7)\times\mu_3}\left(\yng(2,1), \yng(2,1,1), \yng(1)\right) &= \left[\left(\yng(1,1), \yng(2,1,1), \yng(1)\right) \boxtimes 0\right] \\
&\oplus \left[\left(\yng(2), \yng(2,1,1), \yng(1)\right) \boxtimes 0\right] \\
&\oplus \left[\left(\yng(2,1), \yng(2,1), \yng(1)\right) \boxtimes 1\right] \\
&\oplus \left[\left(\yng(2,1), \yng(1,1,1), \yng(1)\right) \boxtimes 1\right] \\
&\oplus \left[\left(\yng(2,1), \yng(2,1,1), -\right) \boxtimes 2\right]
\end{aligned}
$$

Furthermore, it follows that induction from this subgroup consists of the sum of all of the (legal) ways to add a cell to the component of the r-tuple corresponding to that power of ξ.

Example 4.1.8. Taking the components $\left(\yng(2,1), \yng(2,1), \yng(1)\right)\boxtimes 1$ and $\left(\yng(2,1), \yng(2,1,1), -\right)\boxtimes 2$ from the previous example we have that

$$
\mathrm{Ind}^{G(3,1,8)}_{G(3,1,7)\times\mu_3}\left[\left(\yng(2,1), \yng(2,1), \yng(1)\right) \boxtimes 1\right] = \left(\yng(2,1), \yng(2,1,1), \yng(1)\right) \oplus \left(\yng(2,1), \yng(2,2), \yng(1)\right) \oplus \left(\yng(2,1), \yng(3,1), \yng(1)\right),
$$

since there are three legal ways of adding a cell to the second component. On the other hand,

$$\mathrm{Ind}_{G(3,1,7)\times\mu_3}^{G(3,1,8)}\left[\left(\yng(2,1),\yng(2,1,1),-\right)\boxtimes 2\right]=\left(\yng(2,1),\yng(2,1,1),\yng(1)\right),$$

since there is only one way to add a cell to the third component.

4.2 The McKay Quiver of $G(r,1,n)$

We know that the McKay quiver of $G(r,1,n)$ will be a connected graph with a node for each irreducible representation of G. Connectedness is guaranteed by the fact that the action $-\otimes V_{std}$ is always faithful [10]. So understanding the McKay graph comes down to describing when there is an arrow from one node to another.

Theorem 4.2.1. *There will be an arrow from one node, V_α, to another node, V_β, if the $r-$tuple of Young diagrams β can be obtained from α by deleting a cell from position i in α and then adding a cell to position $i+1 \mod r$.*

Proof. Let $W=[(\square\square\cdots\square,-,\cdots,-)\boxtimes 1]$ be a representation of $H=G(r,1,n-1)\times\mu_r$ and $U=V_\alpha$ be an irreducible representation of $G=G(r,1,n)$. Notice that the Young diagram part of W is associated to the trivial representation of $G(r,1,n-1)$.

Now, applying Frobenius reciprocity [see Lemma 1.5.6],

$$\begin{aligned}\mathrm{Ind}_H^G\left\{\mathrm{Res}_H^G V_\alpha \otimes [(\square\square\cdots\square, -, \cdots, -) \boxtimes 1]\right\} &= V_\alpha \otimes \mathrm{Ind}_H^G [(\square\square\cdots\square, -, \cdots, -) \boxtimes 1] \\ &= V_\alpha \otimes (\square\square\cdots\square, \square, -, \cdots, -) \\ &= V_\alpha \otimes V_{std},\end{aligned}$$

which is exactly what we want - so this comes down to understanding what is happening in the left-hand side of the equation.

We know from the above discussion that $\mathrm{Res}_H^G V_\alpha$ can be understood as a direct sum of r-tuples of young diagrams, each obtained by deleting exactly one cell from V_α together with a power of ξ, i, that records from which component the cell was deleted.

$\mathrm{Res}_H^G V_\alpha \otimes W$ can then be understood by fixing the Young diagrams in $\mathrm{Res}_H^G V_\alpha$ (since the Young diagram in W corresponds to the trivial representation) and increasing the power of ξ in each summand of $\mathrm{Res}_H^G V_\alpha$ by 1 (modulo r).

Inducing on the resulting sum will consist of adding a cell in each summand to the component exactly one step to the right (cyclically) of the component from which the cell was originally deleted. That is, the cell has moved from one component to the component directly to the right.

This is exactly the statement of the theorem. $\square$

We will illustrate the proof with an example.

Example 4.2.2. Consider $\alpha = \left(\ydiagram{3,1}, \ydiagram{1}, -\right)$, an irreducible representation of $G(3,1,5)$. The goal is to find all of the targets of arrows with this node as a source. Note that V_{std} corresponds to $(\ydiagram{4}, \ydiagram{1}, -)$ in this case.

Letting $G = G(3,1,5)$ and $H = G(3,1,4) \times \mu_3$, we have that

$$\begin{aligned} \mathrm{Res}_H^G V_\alpha &= [(\ydiagram{3}, \ydiagram{1}, -) \boxtimes 0] \\ &\oplus \left[\left(\ydiagram{2,1}, \ydiagram{1}, -\right) \boxtimes 0\right] \\ &\oplus \left[\left(\ydiagram{3,1}, -, -\right) \boxtimes 1\right]. \end{aligned}$$

Now, the tensor product with $[(\ydiagram{4}, -, -) \boxtimes 1]$ yields

$$\begin{aligned} \mathrm{Res}_H^G V_\alpha \otimes [(\ydiagram{4}, -, -) \boxtimes 1] &= [(\ydiagram{3}, \ydiagram{1}, -) \boxtimes 1] \\ &\oplus \left[\left(\ydiagram{2,1}, \ydiagram{1}, -\right) \boxtimes 1\right] \\ &\oplus \left[\left(\ydiagram{3,1}, -, -\right) \boxtimes 2\right]. \end{aligned}$$

Then induction yields

$$
\begin{aligned}
\mathrm{Ind}_H^G \left\{ \mathrm{Res}_H^G V_\alpha \otimes [(\ydiagram{4}, -, -) \boxtimes 1] \right\} &= \left(\ydiagram{3,1}, \ydiagram{1}, - \right) \otimes (\ydiagram{4}, \ydiagram{1}, -) \\
&= (\ydiagram{3}, \ydiagram{2}, -) \oplus \left(\ydiagram{3}, \ydiagram{1,1}, - \right) \\
&\oplus \left(\ydiagram{2,1}, \ydiagram{2}, - \right) \oplus \left(\ydiagram{2,1}, \ydiagram{1,1}, - \right) \\
&\oplus \left(\ydiagram{3,1}, -, \ydiagram{1} \right).
\end{aligned}
$$

In this example, we can clearly see the available cells are each moving exactly one component to the right and being placed in a way that creates a legal Young diagram.

Chapter 5

The Representations of $G(r, m, n)$

In this chapter, we will describe the irreducible representations of $G(r, m, n)$ as well as how to induce back from $G(r, m, n)$ to $G(r, 1, n)$. This, together with Frobenius reciprocity will allow us to prove our description of the McKay quiver in the next chapter. For a more complete treatment of these complex reflection groups and their irreducible representations, see [4] and [2].

5.1 The Groups $G(r, m, n)$

The groups $G(r, m, n)$ can be thought of as subgroups of the groups $G(r, 1, n)$. The parameters r, m, and n are all positive integers together with the requirement that $m|r$. For our purposes, we will require that $r > 1$ and we will be ignoring the somewhat pathological group $G(2,2,2)$ (see [4]). As with the groups $G(r,1,n)$, the groups $G(r,m,n)$ have a natural presentation as matrix groups of $n \times n$ matrices with factorizations,

$$M = P \cdot D,$$

where P is an $n \times n$ permutation matrix and D is an $n \times n$ diagonal matrix with r^{th}-roots of unity. The groups $G(r,m,n)$ have the added constraint that the diagonal matrix in the factorization satisfies

$$(\det D)^{\frac{r}{m}} = 1.$$

This leads us to the following defining proposition.

Proposition 5.1.1. *The groups $G(r,1,n)$ and $G(r,m,n)$ satisfy the following short exact sequence*

$$1 \longrightarrow G(r,m,n) \longrightarrow G(r,1,n) \longrightarrow \mu_m \longrightarrow 1,$$

where μ_m is the cyclic group of order m and the surjective map is given by

$$\phi(M) = \phi(P \cdot D) = (\det D)^{\frac{r}{m}}.$$

Proof. We need only check that the surjection is a well-defined homomorphism, but this follows by the wreath-product structure in $G(r,1,n)$, as we show below.

Let $\phi : G(r,1,n) \longrightarrow \mu_m$ be defined as above and consider the the elements $M = P \cdot D$ and $N = S \cdot E$ from $G(r,1,n)$, where P and S are permutation matrices and D and E are diagonal matrices of $r^{th}-$roots of unity. Then

$$\begin{aligned}
\phi(M \cdot N) &= \phi((PD) \cdot (SE)) \\
&= \phi((P(S \cdot S^{-1})) \cdot (D \cdot (SE))) \\
&= \phi((PS) \cdot ((S^{-1}DS) \cdot E) \\
&= \det((S^{-1}DS) \cdot E)^{\frac{r}{m}} \\
&= \det(S)^{\frac{-r}{m}} \cdot \det(D)^{\frac{r}{m}} \cdot \det(S)^{\frac{r}{m}} \cdot \det(E)^{\frac{-r}{m}} \\
&= \det(D)^{\frac{r}{m}} \cdot \det(E)^{\frac{r}{m}} \\
&= \phi(M) \cdot \phi(N).
\end{aligned}$$

It is clear, then, that $G(r,m,n) = \ker \phi$ and this is an exact sequence, as claimed. $\square$

Remark 5.1.2. This proposition has some useful consequences, namely that $G(r,m,n)$ is a *normal* subgroup of $G(r,1,n)$ and that the cokernel, $G(r,1,n)/G(r,m,n)$ is a cyclic group. This will help us a great deal as we begin to describe the irreducible representations of $G(r,m,n)$.

5.2 Representation Theory Revisited

We will need a few results for restricting representations of G to a normal subgroup H. Our reference for this section is Grove [7].

Definition 5.2.1. Given $H \trianglelefteq G$, φ a character of H, and $x \in G$, conjugation of φ by x is defined as

$$\varphi^x(h) = \varphi(xhx^{-1}), \text{ for all } h \in H.$$

Proposition 5.2.2. *Let $H \trianglelefteq G$, then G acts by conjugation as a permutation on the irreducible representations of H.*

Definition 5.2.3. The stabilizer of φ is the subgroup of G defined by

$$\mathrm{Stab}_G(\varphi) = \{x \in G \mid \varphi^x = \varphi\}.$$

This is sometimes referred to as $I_G(\varphi)$, the inertia group of φ in G.

Theorem 5.2.4 (Clifford's Theorem)**.** *Suppose that H is a normal subgroup of G and let χ be an irreducible representation of G and φ be an irreducible constituent of χ_H. Then*

$$\chi_H = e \sum_{i=1}^{t} \varphi^{x_i},$$

where $\{x_1, \cdots, x_t\}$ is a transversal for the stabilizer of φ in G and

$$e = [\chi_H, \varphi]_H,$$

the multiplicity of φ in χ_H.

There are many generalizations of Clifford's Theorem: in our case, the cokernel G/H being a cyclic group yields the following proposition, Proposition

6.1 from Stembridge [11].

Proposition 5.2.5. *Let $H \trianglelefteq G$ such that G/H is a cyclic group. Further, let χ be an irreducible character of G with $\chi_H = e\sum_{i=1}^{t}\varphi^{x_i}$. Then the following hold:*

- *G acts transitively on the irreducible constituents of χ_H;*
- *the consituents of χ_H are each inequivalent irreducible characters of H;*
- *φ^G is the sum of the distinct irreducible representations $\chi \otimes \lambda$ where λ is any linear representation of G/H.*

Corollary 5.2.6. *If $H \trianglelefteq G$ such that G/H is a cyclic group, then the multiplicity of the constituents of χ_H in Theorem 5.2.4 are all one. That is, if φ is an irreducible constituent of χ_H, then we can simplify the second statement in Clifford's Theorem to*

$$\chi_H = \sum_{i=1}^{t}\varphi^{x_i}.$$

Remark 5.2.7. A few things to note here are:

- t in Theorem 5.2.4 is exactly the index of the stabilizer in G, $[G : I_G(\varphi)]$;
- if $t = 1$, then this means that the irreducible representation χ remains irreducible after restriction;

- if $t \neq 1$, then this means that χ, after restriction, splits into t irreducible representations of equal dimension which are all conjugate to each other in G.

Unfortunately, this requires us to already have access to the irreducible representations of H which, in our case, we do not. Thus, we need to invoke the following corollaries which appear in the exercises at the end of Section 6.2 in [7].

Proposition 5.2.8. *Let $H \trianglelefteq G$ such that G/H is abelian and let χ and φ be irreducible representations of G. We write $\widehat{G/H}$ to mean the set of irreducible representations of G/H. Here,* $\mathrm{Orb}_{\widehat{G/H}}(\varphi)$ *indicates the set of irreducible representations of G that are conjugate to φ over H.*

- $\chi_H = \varphi_H$ *iff* $\chi \in \mathrm{Orb}_{\widehat{G/H}}(\varphi)$*;*
- $[\chi_H, \chi_H] = |\mathrm{Stab}_{\widehat{G/H}}(\chi)|$*;*
- χ_H *is irreducible in H iff* $|\mathrm{Stab}_{\widehat{G/H}}(\chi)| = 1$.

Proof. First note that, since G/H is abelian, we will have that the representations, $\widehat{G/H}$, are all *linear* representations. Furthermore, we know that a tensor with a linear representation will also act as a permutation of the irreducible representations of G. Hence, in this case, we can think of conjugation by elements of G as tensor products with the linear representations of G/H.

For the first statement, suppose that $\chi \in \mathrm{Orb}_{\widehat{G/H}}(\varphi)$. This can only be true if there exists a $\lambda \in \widehat{G/H}$ such that $\chi \otimes \lambda = \varphi$. If φ splits into t irreducibles with multiplicity one (see Corollary 5.2.6), then we have

$$\begin{aligned} t = [\chi \otimes \lambda, (\varphi_H)^G] &= [(\chi \otimes \lambda)_H, \varphi_H] \\ &= [\chi_H \otimes \lambda_H, \varphi_H] \\ &= [\chi_H, \varphi_H] \end{aligned}$$

The first equality follows because φ_H will be made up of t constituents that all induce up to φ. The second equality is by Frobenius reciprocity and the fact that λ_H is the trivial representation in H. Clearly, if $[\chi_H, \varphi_H] = t$, then $\chi_H = \varphi_H$.

For the second statement, note that, by Frobenius reciprocity, we have

$$[\chi_H, \chi_H] = [\chi, (\chi_H)^G].$$

We know that $(\chi_H)^G$ must consist of the direct sum of all irreducible representations that restrict to χ_H, but from the previous corollary, we know that these are all of the irreducibles in $\mathrm{Orb}_{\widehat{G/H}}(\chi)$. Hence,

$$[\chi, (\chi_H)^G] = \sum_i [\chi, \chi \otimes \lambda_i],$$

where λ_i runs over all of the representations of $\widehat{G/H}$. This pairing will be zero unless χ is fixed by $-\otimes\lambda_i$ which yields

$$\sum_i [\chi,\chi\otimes\lambda_i] = |\mathrm{Stab}_{\widehat{G/H}}(\chi)|.$$

The third statement follows immediately from the previous statement where $t = 1$. $\square$

Remark 5.2.9. Roughly, we can understand the set of irreducibles of $G(r,m,n)$ as a quotient of the set of irreducibles of $G(r,1,n)$ by the action of the linear representations of $G(r,1,n)$ that land in the cokernel, μ_m. However, we will have the added complication that if the representation is *fixed* by any elements of μ_m, then the representation will split after restriction. So our next step is to understand the action of the linear representations of $G(r,1,n)$.

5.3 Linear Actions

Recall that the irreducible representations of $G(r,1,n)$ naturally correspond to $r-$tuples of Young diagrams of total size n. In the foregoing discussion let us define ξ to be a primitive $r^{th}-$root of unity. We begin with some locally useful notation.

Notation 5.3.1. For this section, let $0 \le \ell < r-1$ and λ_ℓ represent the one-dimensional representation corresponding to the Young diagram

$$\lambda_\ell = (-,-,\cdots,-,\square\square\cdots\square,-,\cdots,-)\,,$$

where the (non-empty) trivial Young diagram appears in the $(\ell+1)^{th}$ position of the $r-$tuple.

So, for instance, λ_0 would be the trivial representation of $G(r,1,n)$.

Proposition 5.3.2. *The linear representations of $G(r,1,n)$, λ_ℓ, described above, act on the representations of $G(r,1,n)$ by cycling the diagrams through the $r-$tuple in steps of size ℓ.*

We can prove the proposition by analyzing how the group generators act on the representation before and after being acted upon by $-\otimes\lambda_\ell$.

Proof. Given a representation V_α corresponding to the $r-$tuple of shape α, recall that this representation is spanned by formal linear combinations of Young tableaux, $\mathbf{t}_p$, all of which have shape α.

Recall from Section 3.2 that the generators act on these $\mathbf{t}$'s by

$$s_k\mathbf{t}_p = \begin{cases} \mathbf{t}_p & \text{if } k \text{ and } k-1 \text{ are in the same row,} \\ -\mathbf{t}_p & \text{if } k \text{ and } k-1 \text{ are in the same column,} \\ \mathbf{t}_q & \text{otherwise,} \end{cases}$$

for $k > 1$ and where $\mathbf{t}_q$ is the tableaux $\mathbf{t}_p$ with $k-1$ and k exchanged (if possible).

Furthermore, recall also that

$$t_k : \mathbf{t}_p \longmapsto \xi^{i-1}\mathbf{t}_p,$$

where the number k appears in the i^{th} position of the tableau $\mathbf{t}_p$.

Now consider the representation λ_ℓ: V_{λ_ℓ} will be a one-dimensional vector space spanned by the tableau $\mathbf{s}$ where

$$\mathbf{s}_\ell = \left(-, -, \cdots, -, \boxed{1}\boxed{2}\cdots\boxed{n}, -, \cdots, -\right).$$

The action of the generators on $\mathbf{s}_\ell$ can then be calculated as follows

$$\begin{aligned} s_k(\mathbf{s}_\ell) &= \mathbf{s}_\ell, \qquad \text{if } k > 1, \\ t_k(\mathbf{s}_\ell) &= \xi^\ell \cdot \mathbf{s}_\ell, \qquad \forall k. \end{aligned}$$

We can now examine the action of the generators on the vector space

$$V_\alpha \otimes V_{\lambda_\ell} = \mathrm{Span}\{(\mathbf{t}_p) \otimes \mathbf{s}_\ell\}_p.$$

That is

$$s_k(\mathbf{t}_p \otimes \mathbf{s}_\ell) = s_k(\mathbf{t}_p) \otimes s_k(\mathbf{s}_\ell) = \begin{cases} \mathbf{t}_p \otimes \mathbf{s}_\ell & \text{if } k \text{ and } k-1 \text{ are in the same row,} \\ -\mathbf{t}_p \otimes \mathbf{s}_\ell & \text{if } k \text{ and } k-1 \text{ are in the same column,} \\ \mathbf{t}_q \otimes \mathbf{s}_\ell & \text{otherwise.} \end{cases}$$

And:

$$\begin{aligned} t_k(\mathbf{t}_p \otimes \mathbf{s}_\ell) &= t_k(\mathbf{t}_p) \otimes t_k(\mathbf{s}_\ell) \\ &= (\xi^{i-1}\mathbf{t}_p) \otimes (\xi^\ell \cdot \mathbf{s}_\ell) \\ &= \xi^{(i+\ell)-1} \cdot (\mathbf{t}_p \otimes \mathbf{s}_\ell). \end{aligned}$$

This all shows that the Young diagram associated to $\alpha \otimes \lambda_\ell$ has the same *shape* as α except that all of the cells are now located exactly ℓ steps to the right. $\square$

Example 5.3.3. Let $\alpha = \left(\ydiagram{2,1}, -, \ydiagram{1,1}, \ydiagram{1}\right)$ be a representation of $G(4,1,7)$ then the representation λ_2 will be of the form

$$\lambda_2 = \left(-, -, \ydiagram{7}, -\right).$$

and we will have that

$$\alpha \otimes \lambda_2 = \left(\ydiagram{1,1}, \ydiagram{1}, \ydiagram{2,1}, -\right).$$

5.4 Representations of $G(r,m,n)$

Now, with the consequences of Clifford's Theorem at the end of Section 5.2 and the previous section's work, we are now ready to describe the irreducible representations of $G(r,m,n)$.

Proposition 5.4.1. *The irreducible representations of $G(r,m,n)$ are either equivalence classes of the irreducible representations of $G(r,1,n)$ under the action of the group generated by the linear representation $\lambda_{\frac{r}{m}}$ or constituents of an irreducible representation of $G(r,1,n)$ which splits after restriction. Let α be an irreducible representation of $G(r,1,n)$. Then:*

- *if $|\mathrm{Stab}_{\lambda_{\frac{r}{m}}}(\alpha)| = 1$, then α remains irreducible after restriction;*

- *if* $|\mathrm{Stab}_{\lambda_{\frac{r}{m}}}(\alpha)| = d > 1$*, then* α *will split into* d *distinct components.*

Proof. The proposition is clear from the work we have done so far, but perhaps some clarification is in order.

First, we can see that this action, $- \otimes \lambda_{\frac{r}{m}}$ is really the correct one to yield the exact sequence

$$1 \longrightarrow G(r, m, n) \longrightarrow G(r, 1, n) \longrightarrow \mu_m \longrightarrow 1.$$

We can then think of successive actions by $\lambda_{\frac{r}{m}}$ as forming a cyclic group which acts on the irreducible representations of $G(r, 1, n)$. This cyclic group is μ_m.

If the stabilizer of α is of order one, then clearly each rotation of the diagrams in α is distinct for rotations of size $\frac{r}{m}$. This means that α will restrict to an irreducible representation of $G(r, m, n)$ and so does every other irreducible in the orbit of α under the action of μ_m. That is

$$\mathrm{Res} V_\alpha = \mathrm{Res} V_{\alpha \otimes \lambda^{\otimes i}}, \, \forall i \in \{0, 1, \cdots, m-1\}.$$

Note also that the restriction will have the same dimension as α.

If the stabilizer is not of order one, then some rotation of the diagrams of α is exactly equal to α itself. That is, the representation α has some rotational symmetry under the action of μ_m. That means that α, after restriction, will split into d irreducible constituents where $d = |\mathrm{Stab}_{\mu_m}(\alpha)|$. It is also clear that the stabilizer is a cyclic subgroup of μ_m and that $d|m$. Note that, since restriction preserves dimension, we will have that the constituents of the restriction of α will have dimension

$$\frac{\dim V_\alpha}{d}.$$

□

Remark 5.4.2 (Description of the Irreducible Constituents)**.** It would be convenient to have a way of somehow preserving the $r-$tuple description of the representations, for the purposes of computing dimension and to explicitly compute the McKay quiver of a given $G(r, m, n)$. To that end, consider the following presentation of the irreducibles: Case 1 is for $G(r, r, n)$ and Case 2 is for any $G(r, m, n)$:

Case 1 (when $r = m$): What we need is to express an $r-$tuple in a way that remembers the *order* of the tuple, but that 'forgets' which diagram is *first*. This is naturally accomplished by drawing the $r-$tuple around the outside of a circle. Drawing them this way, it becomes obvious which irreducible representations of $G(r, 1, n)$ belong in the same equivalence class after restriction. It is also clear when an irreducible representation will split: if the $r-$tuple has rotational symmetry, then the irreducible will split after restriction into a number of components equal to the number of symmetries of that figure.
Case 2 (when merely $m|r$): This is similar to the previous case except, in-

stead of having individual Young diagrams along the perimeter of the circle, there will be m $\frac{r}{m}-$tuples of Young diagrams around the perimeter of the circle. These tuples are constructed in the following way: the first $\frac{r}{m}$ Young diagrams form the first $\frac{r}{m}-$tuple, then the second set of $\frac{r}{m}$ Young diagrams forms the second $\frac{r}{m}-$tuple and so on until all of the Young diagrams have been placed. This construction ensures that the action of $\lambda_{\frac{r}{m}}$ is trivial on the representations after restriction.

Note that the author uses the convention that $r-$tuples are placed around the circle in a clockwise manner.

Note also that this reduces quite nicely to Case 1 when $m = r$.

Preserving the Young diagram presentation will allow us to compute the dimensions just as in $G(r, 1, n)$ with the added complication that figures with rotational symmetries must have their dimension divided by the number of symmetries that that figure possesses.

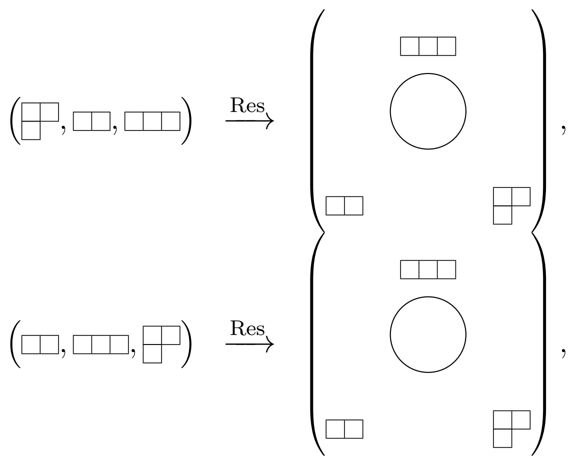
Example 5.4.3 $(r = m)$**.** Consider

$$\alpha = \left(\yng(2,1), \yng(2), \yng(3)\right),$$

and

$$\beta = \left(\yng(2), \yng(3), \yng(2,1)\right),$$

be irreducible representations in $G(3,1,8)$. Then we will have that

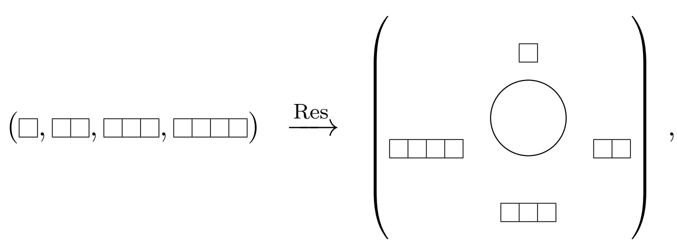


both restrict to the same irreducible representation in $G(3,3,8)$.

Example 5.4.4 $(r|m)$**.** Consider

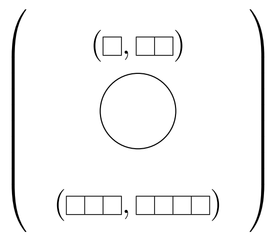
$$\alpha = (\square, \square\square, \square\square\square, \square\square\square\square)\,,$$

an irreducible representation in $G(4,1,10)$. Then



in $G(4,4,10)$ and,

$$(\square, \square\square, \square\square\square, \square\square\square\square) \xrightarrow{\text{Res}}$$



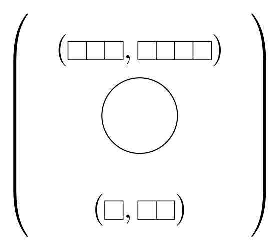
,

in $G(4,2,10)$. Note that the *order* within each $\frac{r}{m}-$tuple is important, but again, not which $\frac{r}{m}-$tuple appears *first*. To illustrate, consider

$$\alpha \otimes \lambda_{\frac{4}{2}} = (\square\square\square, \square\square\square\square, \square, \square\square)\,.$$

By construction, we know that α and $\alpha \otimes \lambda_2$ must have the same restriction to $G(4,2,10)$. Hence

$$\alpha \otimes \lambda_2 = (\square\square\square, \square\square\square\square, \square, \square\square) \xrightarrow{\text{Res}}$$



,

and we can see that $\alpha_H = (\alpha \otimes \lambda_2)_H$, as required.

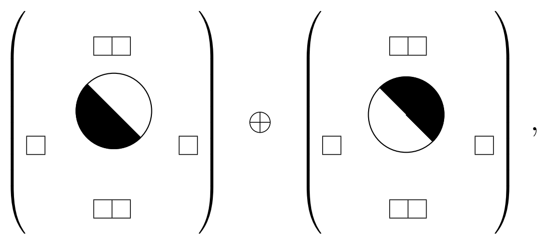
Definition 5.4.5. We will call the *fundamental domain* of an irreducible representation of $G(r,m,n)$ the sector of $\frac{r}{m}-$tuples on the circle that repeat t times around the representation.

Thus, if an irreducible has no symmetry ($t = 1$), then it has only one fundamental domain which consists of the entire circle.

If an irreducible has $t \neq 1$ symmetries, then it has t conjugate representations in $G(r, m, n)$, which we can distinguish by their fundamental domains.

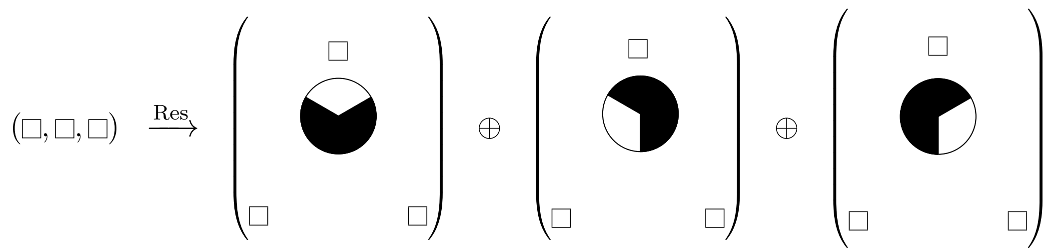
Example 5.4.6. Take $\alpha = (\square\square, \square, \square\square, \square)$ in $G(4, 1, 6)$. Clearly, this figure has an order 2 symmetry and so will split after restriction to $G(4, 4, 6)$ into two irreducibles. These irreducibles are identical except for their fundamental domains

$$(\square\square, \square, \square\square, \square) \xrightarrow{\text{Res}}$$



so that the Young diagrams with a white sector 'beneath' them are in the fundamental domain. Note that we have made a choice here in how we have placed the fundamental domain. It should be clear that the choice of fundamental domain is immaterial, so we will adopt the convention that the first fundamental domain will start under the Young diagram at the top of the figure and run clockwise from there.

Example 5.4.7. Consider $\alpha = (\square, \square, \square)$ in $G(3, 1, 3)$ then its restriction to $G(3, 3, 3)$ splits into three consituents with identical Young diagrams, but distinct fundamental domains



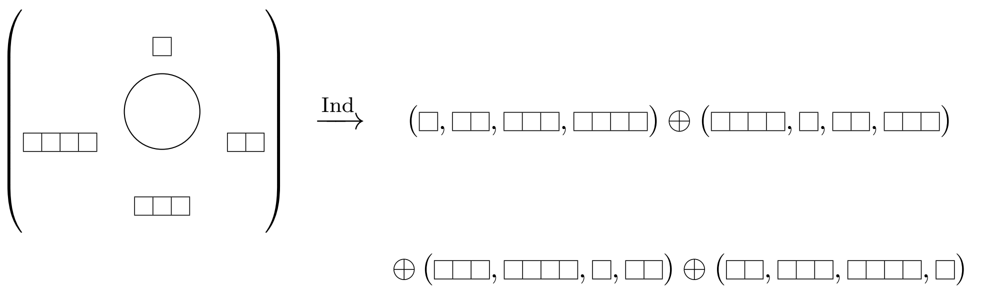
Remark 5.4.8. We think of the fundamental domains described above as highlighting which cells of the m $\frac{r}{m}-$tuples are 'active'. That is, these are the cells that are potentially available for induction and also for the action of $-\otimes V_{std}$ which will be described below and in the following section.

Remark 5.4.9 (Inducing)**.** Induction from $G(r,m,n)$ to $G(r,1,n)$ involves 'unfurling' the diagram in all possible *distinct* ways. Each of these components are related by the linear actions from $G(r,1,n)/G(r,m,n) \simeq \mu_m$: that is, they are all related by rotation of their associated $r-$tuple.

If φ, an irreducible in $G(r,m,n)$, has no rotational symmetry, then every possible unfurling is distinct and so $\text{Ind}V_{\varphi}$ will consist of m distinct $r-$tuples each related to the others by the action of μ_m.

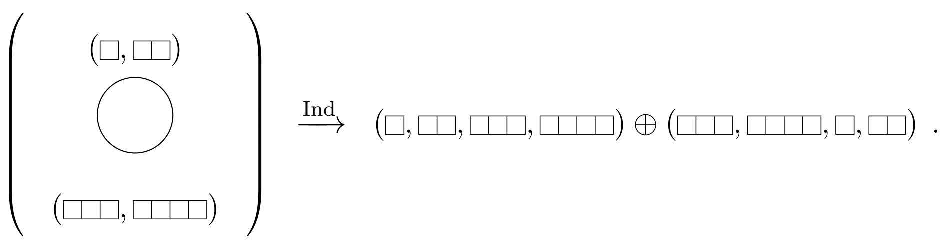
If φ has t rotational symmetries, then $\text{Ind}V_{\varphi}$ will consist of $\frac{m}{t}$ non-isomorphic components. We can think of this as arising from the fact that only the $\frac{r}{m}-$tuples in the fundamental domain of the representation are allowed to be 'first' in our unfurling.

Example 5.4.10. To build on the examples already given, we see that

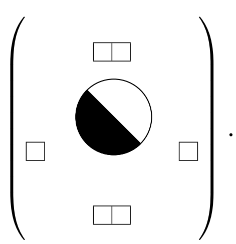


We can clearly see here the orbit of $- \otimes \lambda_1$.

Now, if we are in the case that $m \neq r$ then the unfurlings consist of the direct sum of all possible concatenations of the $\frac{r}{m}$−tuples in clockwise order. So, our example from $G(4, 2, 10)$ yields

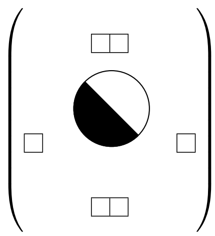


Finally, consider the following irreducible representation in $G(4, 4, 6)$



This irreducible will induce up to two distinct irreducible representations in

$G(4,1,6)$

 $\xrightarrow{\text{Ind}} \; (\square\square, \square, \square\square, \square) \oplus (\square, \square\square, \square, \square\square) \; .$

Note that if all of the cells were 'active', then there would be two more irreducibles in the direct sum: this is the usefulness of the fundamental domain in our notation.

Remark 5.4.11. Notice that the restriction of the standard representation V_{std}, represented by the $r-$tuple $(\square\square\cdots\square, \square, -, \cdots, -)$, to $G(r,m,n)$ is irreducible (*i.e.* this diagram has no rotational symmetries). It should be pointed out that this is where we must exclude the groups $G(2,1,2)$ and $G(2,2,2)$ from our discussion: the standard representation in this case is *not* irreducible after restriction, as is readily observed.

Chapter 6

The McKay Quiver of $G(r, m, n)$

In this section we describe the construction of the McKay quiver of the complex reflection groups, $G(r, m, n)$. It turns out that the proof is relatively simple, given Frobenius reciprocity; the main difficulty is in describing the irreducible representations, which has been accomplished in the previous section.

6.1 The McKay Quiver of $G(r, m, n)$

Before we can prove our description of the McKay quiver, we must first introduce a few lemmas.

Lemma 6.1.1. *Suppose that χ is an irreducible representation of $G(r, 1, n)$ such that $\chi_{G(r,m,n)}$ has $t > 1$ rotational symmetries.Llet φ be any irreducible representation in $G(r, 1, n)$ and let σ be the standard representation in $G(r, 1, n)$.*

If

$$[\chi \otimes \sigma,\, \varphi] \neq 0,$$

then $\varphi_{G(r,m,n)}$ *has no rotational symmetry and, so is irreducible in* $G(r, m, n)$.

Proof. Recall that the action $- \otimes V_{std}$ consists of all possible ways to move a single cell exactly one step to the right. It is clear that this action will break any rotational symmetry in χ. □

Lemma 6.1.2. *Suppose that* χ *is an irreducible representation of* $G(r, 1, n)$ *such that* $\chi_{G(r,m,n)}$ *has* t *rotational symmetries. Let* φ *be any irreducible representation in* $G(r, m, n)$ *and let* σ *be the standard representation in* $G(r, 1, n)$. *Then,*

$$t \mid [\chi \otimes \sigma,\, \varphi^{G(r,1,n)}].$$

Proof. If moving a single cell of χ can result in a component of

$$\varphi^G = \sum_{i=1}^{m} \phi \otimes \lambda_i,$$

where ϕ is any consituent of φ^G, then there will be t constituents of $\chi \otimes \sigma$ that are each some distinct rotation of ϕ. All of these rotations appear, so the lemma is proved. □

We are now ready to describe the construction of the McKay quiver of $G(r, m, n)$.

Theorem 6.1.3. *Let* V_α *and* V_β *be irreducible representations of* $G(r, m, n)$. *There will be an arrow with source* V_α *and target* V_β *in the McKay quiver* *of $G(r,m,n)$ if the diagram of β can be obtained from the diagram of α by moving a single cell in the fundamental domain of α clockwise one step.*

Proof. Recall that V_{std} will remain standard after restriction and let σ be the character associated to V_{std}. Given V_χ, irreducible in $G(r,1,n)$, with character χ and let V_φ be an irreducible representation in $G(r,m,n)$ with character φ. We have, by Frobenius reciprocity,

$$\begin{aligned}[\chi \otimes \sigma,\, \varphi^{G(r,1,n)}] &= [(\chi \otimes \sigma)_{G(r,m,n)},\, \varphi] \\ &= [\chi_{G(r,m,n)} \otimes \sigma_{G(r,m,n)},\, \varphi].\end{aligned}$$

Now, the first line represents the McKay quiver in G whereas the last line represents the McKay quiver in H.

Case 1 ($\chi_{G(r,m,n)}$ is irreducible in H)

The statement is immediate from the above calculation, after some thought. φ^G consists of all possible 'unfurlings' of the diagram of φ, so there wil be an arrow from V_{χ_H} to V_φ in McK(H) whenever there is an arrow from V_χ to one of the components of V_{φ^G} in McK(G). We know from our work on the McKay quiver of $G(r,1,n)$ that this will only be the case if the associated $r-$tuple of some a component of V_{φ^G} can be obtained from that of V_χ by moving a single cell one step to the right (cyclically). After restriction, this means that the diagram associated to φ is obtained from that of χ_H by moving a single cell one step clockwise.

Case 2 ($\chi_{G(r,m,n)}$ splits in H)

Suppose that

$$\chi_{G(r,m,n)} = \sum_{i=1}^{t} \vartheta^{x_i},$$

where ϑ is some irreducible constituent of $\chi_{G(r,m,n)}$ as in Clifford's Theorem. Then we have the further calculation

$$\begin{aligned}[\chi \otimes \sigma,\, \varphi^{G(r,1,n)}] &= [\chi_{G(r,m,n)} \otimes \sigma_{G(r,m,n)},\, \varphi] \\ &= \textstyle\sum_{i=1}^{t} [\vartheta^{x_i} \otimes \sigma_{G(r,m,n)},\, \varphi].\end{aligned}$$

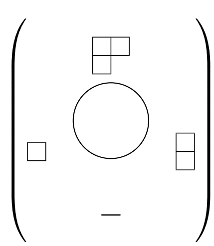
The only way for this equation to hold, given Lemma 6.1.2, is for each set of t arrows indicated by the first pairing to be shared equally among each ϑ^{x_i} in the last pairing. Noting that each ϑ^{x_i} has the same diagram, but a distinct fundamental domain, we can think of this action as moving a single available cell in the fundamental domain of ϑ^{x_i} one step clockwise.

This completes our proof. $\square$

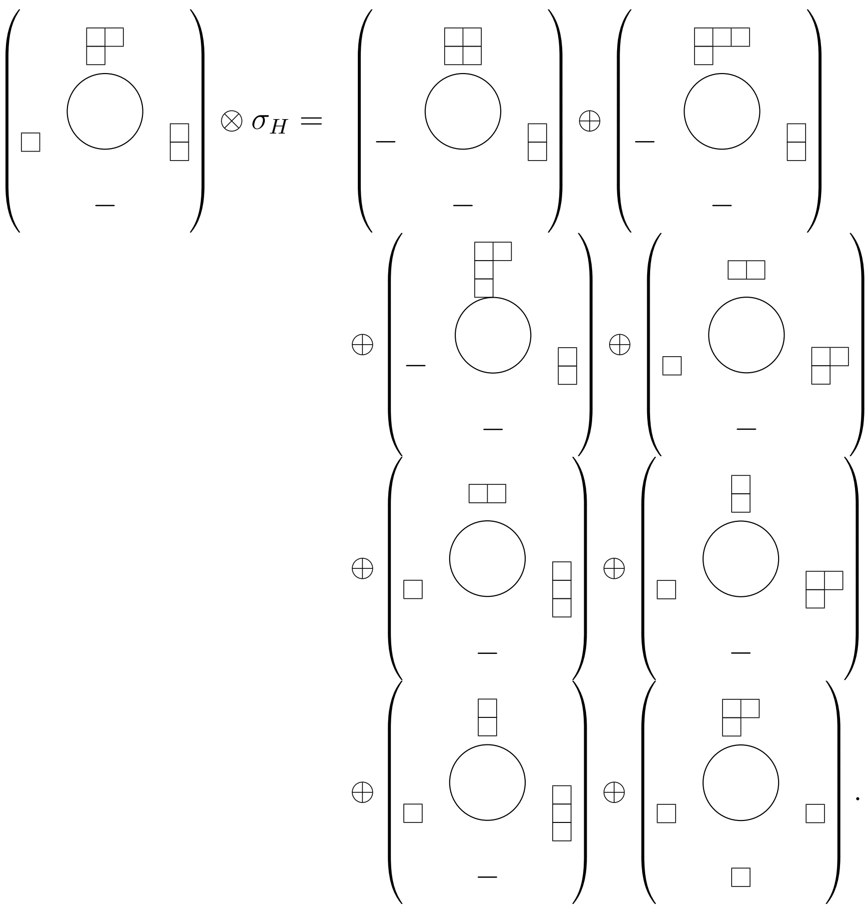
6.2 Examples of $\mathrm{McK}(G(r, m, n))$

We now demonstrate Theorem 6.1.3 with a few examples.

Example 6.2.1. Let

$\alpha =$ 

be an irreducible representation of $G(4,4,6)$. Then we can compute

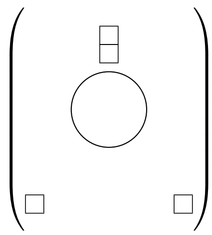


What this computation shows is that there are eight arrows in the McKay quiver of $G(4,4,6)$ with source α. The targets of these arrows are exactly the irreducible representations appearing in the direct sum.

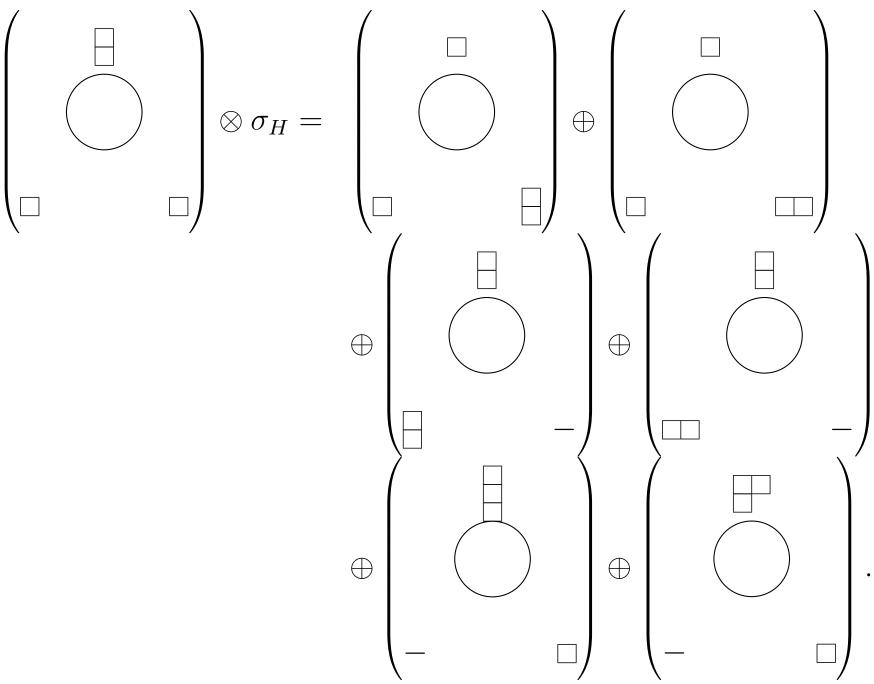
Proposition 6.2.2. *The McKay quiver for $G(r,m,n)$ can have loops or multiple arrows between nodes.*

This assertion is proved by Examples 6.2.3 and 6.2.4.

Example 6.2.3. Consider the irreducible representation

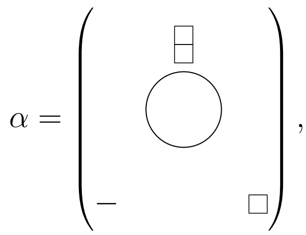
$\alpha =$  ,

in $G(3,3,4)$. We can compute the arrows in $\mathrm{McK}(G(3,3,4))$ with α as a source by

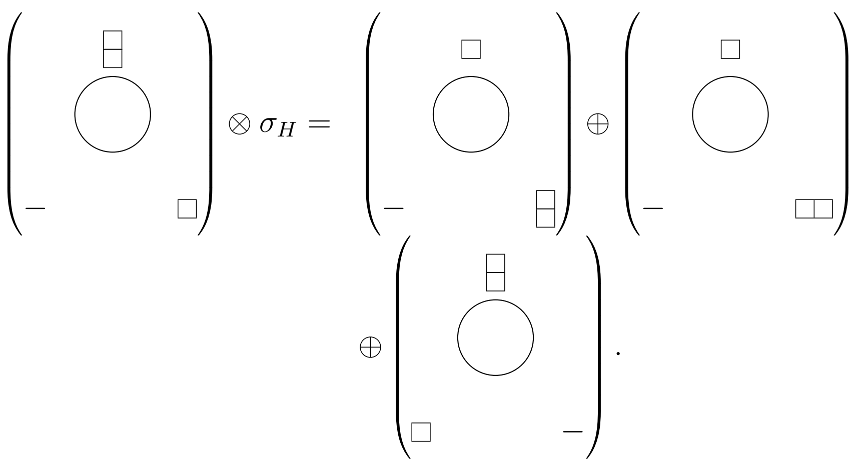


Notice that, given the fact that these diagrams are invariant under rotation, the first summand is equal to α. This corresponds to a loop in the McKay quiver.

Example 6.2.4. Consider the irreducible representation

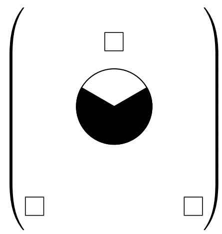


in $G(3,3,3)$. Then computing the arrows in $\mathrm{McK}(G(3,3,3))$ with α as a source

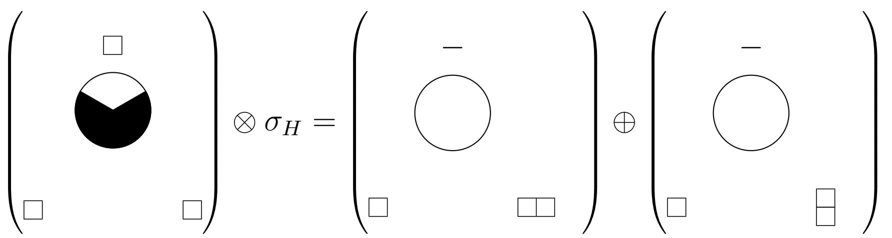


Notice that the first and third summands in the direct sum correspond to the same irreducible representation in $G(3,3,3)$. Hence, there is a double arrow from α to this irreducible in $\mathrm{McK}(G(3,3,3))$.

Example 6.2.5. Let

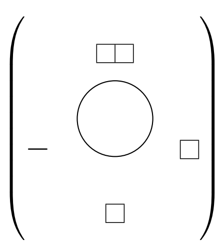
$\alpha =$  ,

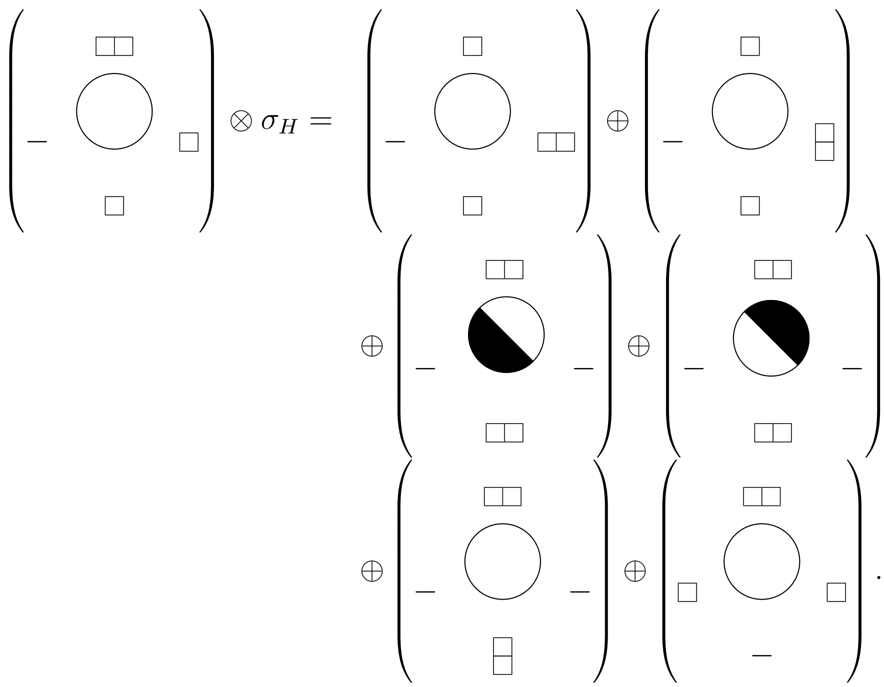
in $G(3,3,3)$. Then we act by $- \otimes \sigma_H$ by



This follows from only the one cell being available (as there is only one available cell in the fundamental domain) and only two ways of constructing a valid Young diagram with this cell in the clockwise-adjacent diagram.

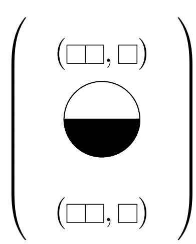
Example 6.2.6. It is clear from the construction that should a target of a representation have any rotational symmetry, then all conjugates of that representation must appear (distinguishable by their fundamental domains). Take, for instance,

$\alpha =$  , a representation of $G(4, 4, 4)$. Then we have

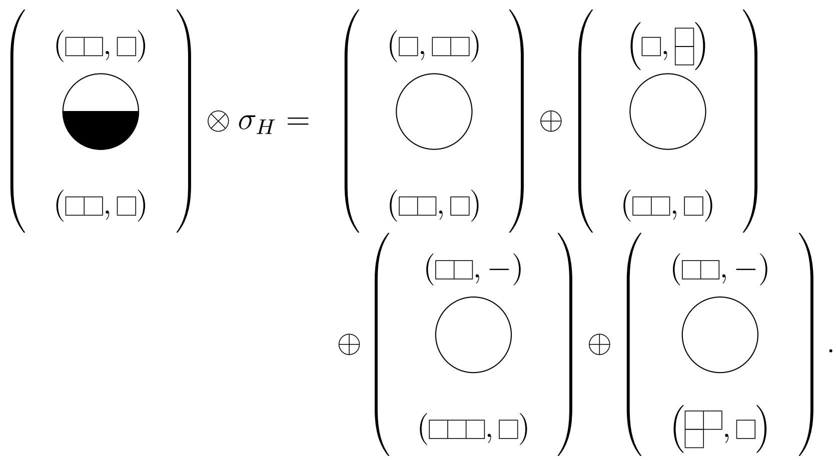


Remark 6.2.7. Note that Lemma 6.1.1 ensures that we never see an arrow between two representations that both have rotational symmetry.

Example 6.2.8. In this example we see the case where $m \neq r$. Let

$\alpha =$ 

be an irreducible representation in $G(4, 2, 6)$. Now, recall that only the cells in the fundamental domain can be available to move, so we have



Notice that the order in which we move cells is compatible with their ordering in $G(4, 1, 6)$.

Conclusions and Further Questions

The material in this thesis closes the question of the McKay quiver of the complex reflection groups, with the exception of the remaining finite list of complex reflection groups - including the Shephard-Todd groups among others - though this is, of course, a finite calculation.

However, as noted in the introduction, the author speculates that the harder problem of computing the relations on the McKay quiver should relate to resolving tensor products with exterior powers of the standard representation of the group. If a combinatorial description of this tensor product could be found - even for S_n - this would be a huge step forward.
The author has some preliminary calculations to this end which are promising, but there is still much work to be done here.

It also seems that the techniques in this thesis could be used to analyse or-

thogonal subgroups of true reflection groups, given their natural description as a restriction of the determinants of the matrices that appear in the overgroup. The reason this should work is clear: the cokernel of the inclusion map will be a cyclic group and so the more specialized version of Clifford's Theorem will apply.
This will, in turn, allow us to compute the McKay quiver of the orthogonal subgroup whenever we can compute the McKay quiver of the overgroup.

In the course of working through this problem, the author also stumbled upon what seems to be an application of the Littlewood-Richardson Rule for multiplying Young diagrams. Namely, the restriction functor from the irreducibles of $G(r, 1, n)$ to those of S_n seems to be an example of the LRR for the multiplication of $r-$many Young diagrams. Fast algorithms for the LRR are in demand and perhaps this example will yield insights into such a thing.

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