Cosmological Perturbation Theory in a Matter-Time Gauge

by

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Abstract

This work examines cosmological perturbations in a Hamiltonian framework with a matter-time gauge. Einstein's field equations are written in a matter-time gauge. The perturbed three-metric of cosmology, its conjugate momentum and the shift are substituted in these equations. The equations of motion of the perturbations to linear order are derived. These equations are expanded in terms of spatial Fourier modes and are then decomposed into scalar, vector and tensor components. After fixing gauges and solving constraints we find that the scalar mode is ultralocal and that the vector modes vanish. We also see that the traceless transverse tensor modes give the known propagation equation for gravitational waves in an expanding, spatially flat, homogeneous and isotropic background.

Dedication

To Dee Saeed for all the adaptations you forced me to make.

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List of Symbols, Nomenclature or Abbreviations

Large scales - Distances of 100 million light years.

$^{(0)}$ - The quantity with this label is a background quantity.

$^{(1)}$ - The quantity with this label is a perturbation.

Canonical Variables - The position and momentum variables.

a - The scale factor for the expansion of the universe; it is dimensionless

H - Hubble parameter, $H = \frac{\dot{a}}{a}$.

ADM - Arnowitt, Deser and Misner.

CMB - Cosmic microwave background.

$FLRW$ - Friedmann–Lemaitre–Robertson–Walker.

GR - Einstein's theory of general relativity.

SFT - Spatial Fourier transform.

SVT - Scalar, vector, tensor.

TRI - Time reparameterization invariant.

Chapter 1

Introduction

Starting a 100-meter race is an event that happens somewhere in space and at some time; it is a spacetime (or space-time) event. By extension crossing the finish line for that race is also a spacetime event. The rule for finding the distance between these two spacetime points is given by the metric tensor g_{ab}. Therefore the metric contains information on the geometry of spacetime. This rule for finding distances in spacetime depends on where one is in space and on what time it is. The metric therefore depends on space and time: $g_{ab}(t, \vec{x})$.

Einstein's general theory of relativity (GR) offers a geometrical explanation of gravity. According to Einstein, the geometry of spacetime - and therefore the metric tensor - can be linked to gravity. Thus gravity, like the metric tensor, is dependent on both space and time. The second finding of GR is best summarized by John Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve" [5]. This is encapsulated in the Einstein field equations:

$$\text{Spacetime Geometry (g)} = \text{Matter-Energy Density} \tag{1.1}$$

where the left hand side is a function of the metric and the right hand side contains information about matter. The metric and the matter have to be solved for together. The remarkable result of the Einstein field equations is then the metric for all space and time along with the dynamics of the matter fields. The Einstein field equations are a set of ten coupled partial differential equations which are generally - by their very nature - hard to solve.

GR is used to study many interesting physical systems and phenomena such as black holes, the perihilion shift in the orbit of Mercury and the large-scale universe[1]. The universe has a spacetime and matter that we can observe. However we can only observe the part of the universe from which we have received light. We use GR to investigate beyond what is observed and - for example - chart the full history of the universe and predict its fate. To our advantage, using GR to study the universe is simplified by symmetries present in the universe on large scales. Since the time of Copernicus, it has been common to presuppose that we are not in a special position in the large-scale universe: that if we were located somewhere else our observations would be unchanged [10]. This is called homogeneity and it implies that

[1] In this thesis, "large-scales" indicate distances of 100 million light years.

the matter densities and the expansion of the universe are not dependent on position. It was also commonly presupposed that there is no special direction in the universe: that our observations about the universe in the $\hat{\imath}$ direction are not different from those in the $\hat{k}$ direction[2]. This is called isotropy. These symmetries greatly simplify the task of using GR to study the universe.

However just like there are small ripples on a pond, there exist fluctuations to the homogeneous and isotropic universe. For example, different points in the universe have different expansion rates and every direction does not stretch equally fast. Therefore the small ripples are not homogeneous and isotropic. One may re-express the total expansion in terms of a background and a perturbation:

$$m = m^{(0)} + m^{(1)} \tag{1.2}$$

where m is the full quantity, $m^{(0)}$ is the background quantity and $m^{(1)}$ is the perturbation. The procedure of breaking up a quantity as such, finding the equations of motion for the background and then studying the behavior of the perturbations on these background solutions is called perturbation theory.

In cosmological perturbation theory, the cosmological spacetime is broken into a background (which is homogeneous and isotropic) and perturbations (which are non-homogeneous and non-isotropic). Similarly the matter fields in the universe are also broken into backgrounds and fluctuations. The space-time and matter fields are then substituted in the Einstein field equations

[2] $(\hat{\imath}, \hat{\jmath}, \hat{k})$ are the basis vectors in Cartesian coordinates

and the perturbed equations to linear order are derived. The standard procedure for doing so has some well-known results, one of which is related to the propagation of gravitational waves.

This thesis explores cosmological perturbation theory in a Hamiltonian framework with time being represented by a matter field. It is checked that this alternative framework gives known results about the propagation of gravitational waves.

In Chapter 2 we start by writing the Einstein equations. Then we find that in the absence of matter, a flat spacetime is the simplest solution to these equations. Following that we study perturbations on this solution. We find two functions in the metric perturbations that propagate as waves. These are gravitational waves in flat spacetime.

In Chapter 3 we review the standard method that is used to study cosmological perturbation theory. The metric and matter are defined in terms of backgrounds and perturbations. The perturbations are then classified according to how they respond to rotations in Fourier space. For each class of perturbations the corresponding metric and matter terms are substituted in the Einstein field equations and the perturbation equations at linear order are studied. We find that the dynamics of one class of perturbations represent the propagating gravitational waves in the universe.

In Chapter 4 the alternative framework for studying cosmological perturbation theory is set up. We consider a system of gravity with a dust field and derive its Hamiltonian formulation. Following that we equate the dust field to time. The resultant equations of motion become the framework to study perturbation theory in GR.

In Chapter 5 we study cosmological perturbation theory in the new framework. The perturbed cosmological spatial metric and its conjugate momentum are used as inputs. The perturbation equations to linear order are classified based on how they respond to rotations in spatial Fourier space. It is noted that the dynamics of one class of perturbations represents the propagating gravitational waves.

Chapter 6 provides a summary and some concluding remarks to the thesis.

Chapter 2

General Relativity and Perturbation Theory

In this chapter first we review Einstein's equations that relate the geometry of spacetime to matter. In Section 2.2 we check that a flat spacetime - the Minkowski metric - is the simplest solution to the Einstein equations in the absence of matter. In Section 2.4 we introduce perturbations to this background solution and expand the Einstein equations to linear order in the perturbations. We use two different routes to study the perturbation equations [2][7][10][19]. Through both routes we get two degrees of freedom in the metric perturbation that propagate as waves. These are the gravitational waves.

2.1 Einstein Field Equations

The Einstein equations relate the geometry of spacetime to matter. The geometry of spacetime is represented by the metric $g_{ab}(t, \vec{x})$ which is a symmetric tensor[1]. Recall that spacetime is curved by matter. However, to find how each individual matter particle - in (for example) clusters of galaxies or inside of a black hole - affects spacetime is tedious. It is much more sensible to check how matter as a whole entity affects spacetime. Therefore we use a matter distribution instead. We denote this by M and this is also a function of space and time. The dynamical variables in GR are therefore g_{ab} and M.

The Einstein equations are[2]:

$$G_{ab} = 8\pi G T_{ab}. \tag{2.1}$$

1. The right hand side of the Einstein equation has the stress-energy tensor T_{ab}. It is a symmetric tensor that encodes the energy-representing quantities of the matter distribution M. The form of the stress-energy tensor is determined by the matter distribution it represents. For example if there is no matter then the stress-energy tensor is zero:

$$T_{ab} = 0. \tag{2.2}$$

[1]The inverse metric is denoted by g^{ab} and it satisfies the relation $g^{ab}g_{bc} = \delta^a_c$, where δ^a_c is the Kronecker delta. The metric and its inverse are used to lower and raise indices on tensors.

[2]G is Newton's gravitational constant. G = $6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$.

The stress-energy tensor can be used to represent a matter distribution that behaves like a perfect fluid. A perfect fluid is one that can be completely specified by its rest-frame four velocity u^a, energy density ρ and pressure p[3,4] [2] [10]. The pressure of the perfect fluid is the same in all directions. We explain such a pressure through an example. Consider water - a perfect fluid - at rest in a plastic bottle. We note that the pressure the water exerts on the bottle can be given by the interactions of water molecules with the surface of the bottle[5]. Since we do not notice bulging of the bottle surface in any preferred direction we can conclude that the pressure exerted by the water is the same in all directions. If a matter distribution behaves like a perfect fluid then we can express M in terms of its rest frame four-velocity u^a, energy density ρ and pressure p. Its stress-energy tensor is:

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}. \tag{2.3}$$

The stress-energy tensor represents energy and momentum of the system. Therefore it should follow the conservation law that in an infinitesimal volume, changes in energy are entirely dependent on the

[3]The four vector representing position in spacetime is: $x^a = [t, \vec{x}]^T$, where $\vec{x}$ is the position vector in space. The superscript T represents the transpose of the quantity. Therefore x^a is a column vector. The four velocity is $u^a = [1, \vec{v}]^T$ where $\vec{v}$ is the three-velocity.

[4]A rest-frame is one in which the fluid has no three-velocity $\vec{v}$.

[5]Similarly the pressure a matter distribution exerts on a surface is given by the interactions of its composite particles with that surface.

flow of energy in and out of that volume. Associated with the energy is a density ρ. We can represent the movement of energy as a current $\vec{J}$. The change in energy because of its movement through the volume element is given by $\oint \vec{J}.\vec{dS}$ where $\vec{dS}$ is an infinitesimal surface area element. Consequently we can write the equation:

$$\oint \vec{J} \cdot d\vec{S} + \frac{d}{dt} \int \rho dV = 0 \tag{2.4}$$

where V represents the volume. The above in differential form is[6]:

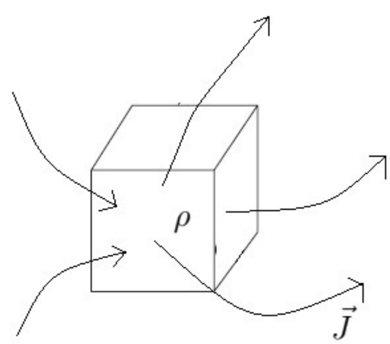

Figure 2.1: Flow of Energy through a Cube. [7]

$$J^i_{,i} + \dot{\rho} = 0. \tag{2.5}$$

[6]A comma represents a partial derivative. Consider the function $F(t, x, y, z)$. $F_{,x}$ represents the partial derivative of that function with respect to x. We can also indicate this as $\partial_x F$.

This is the conservation law in mathematical form and this equation is called the continuity equation. To generalize this to the stress-energy tensor in curved spacetime we write:

$$\nabla^a T_{ab} = 0 \tag{2.6}$$

where ∇_a is the covariant derivative and its action on an arbitrary tensor S^b_c is defined as[7]:

$$\nabla_a S^b_c = S^b_{c,a} + \Gamma^b_{ae} S^e_c - \Gamma^e_{ac} S^b_e \tag{2.7}$$

where Γ^a_{bc} are the Christoffel symbols. They are prescribed as:

$$\Gamma^a_{bc} = \frac{g^{ad}}{2}[g_{db,c} + g_{cb,d} - g_{bc,d}]. \tag{2.8}$$

Therefore equation (2.6) is the continuity equation for the stress-energy tensor generalized to a curved spacetime.

2. The left hand side of equation (2.1) has the Einstein tensor and it encodes information on how spacetime is curved. The Einstein tensor is prescribed as:

$$G_{ab} = R_{ab} - \frac{R g_{ab}}{2} \tag{2.9}$$

[7]It is worth noting that the covariant derivative of S^b_c is not the same as the covariant derivative of S_{bc}.

where R_{ab} is the Ricci curvature tensor and R is its trace[8]. The Ricci tensor is prescribed as:

$$R_{ab} = \Gamma^{c}_{ab,c} - \Gamma^{c}_{ca,b} + \Gamma^{f}_{ab}\Gamma^{c}_{cf} - \Gamma^{f}_{ac}\Gamma^{c}_{bf}. \tag{2.10}$$

Since R_{ab} contains information on how spacetime is curved we can use it for the left hand side of the Einstein equations. However the contracted Bianchi identities:

$$\nabla^{a} R_{ab} = \frac{\nabla_{b} R}{2} \tag{2.11}$$

indicate that the covariant divergence of the Ricci tensor is not zero[9]. Recall that the stress-energy tensor has zero covariant divergence. Hence the Einstein equations will be inconsistent if we use the Ricci tensor for the left hand side. We rearrange the contracted Bianchi identities in equation (2.11) to get:

$$\nabla^{a}(R_{ab} - \frac{g_{ab}R}{2}) = 0. \tag{2.12}$$

The quantity in brackets is the Einstein tensor and it has zero covariant divergence. The Einstein equations are now consistent.

The Einstein tensor encodes information on the curvature of spacetime, contains second order partial derivatives of the metric, has a vanishing

[8] The trace of the Ricci tensor is calculated by raising one index with the inverse metric g^{ab}. Therefore $R = g^{ab}R_{ab}$.

[9] Covariant divergence is the generalization of ordinary divergence to curved spacetime.

covariant divergence and is symmetric. It goes on the left hand side of the Einstein equations.

It may seem that the Einstein equations are a set of ten partial differential equations. This is because the tensors G_{ab} and T_{ab} are symmetric 4×4 tensors. However the conservation law accounts for four differential identities. Therefore there are actually $10 - 4 = 6$ independent partial differential equations.

2.2 Simplest Solution to Einstein's Equations

In this section we solve the Einstein equations to find the simplest metric that corresponds to a system without matter. Absence of matter corresponds to $T_{ab} = 0$. Therefore the Einstein equations simplify to:

$$G_{ab} = 0. \tag{2.13}$$

If G_{ab} equals zero then so should its trace G. It can be checked that in general:

$$G = -R. \tag{2.14}$$

Therefore the trace of the Ricci tensor is also zero. Consequently the Einstein equations are:

$$R_{ab} = 0. \tag{2.15}$$

We recall that the Ricci tensor is a function of the Christoffel symbols which contain first order partial derivatives of the metric. The simplest way for the Ricci tensor to equal zero is if the metric functions were all constants. This is a solution. We further note that a physically meaningful solution is the Minkowski metric, which in (t, x, y, z) coordinates is:

$$\eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.16}$$

It is physically meaningful because it is the metric for flat spacetime which corresponds to an absence of gravity. The interval between two points in a flat spacetime expressed in (t, x, y, z) coordinates is:

$$ds^2 = -dt^2 + e_{ab}dx^a dx^b \tag{2.17}$$

where e_{ab} is the Euclidean metric. In Cartesian coordinates the Euclidean metric is the identity matrix.

Therefore the simplest, physically meaningful solution of the Einstein equation for a system without matter is a flat spacetime.

2.3 Perturbation Theory

Perturbation theory is the study of small fluctuations on a background solution[10]. We consider the analogy of ripples on a pond. We first liken the surface of the pond as the background solution. We then perturb the background solution by creating ripples on the surface. The study of these ripples is perturbation theory. In this thesis we will study perturbations to the background solutions in GR.

The first step in perturbation theory is to break the quantity of interest in a background quantity and a perturbation. For example we break the metric as:

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} \tag{2.18}$$

where g_{ab} is the full metric, $g_{ab}^{(0)}$ is the background metric and $g_{ab}^{(1)}$ represents the perturbation to that background. We also break the stress-energy tensor in a similar way. Next we substitute the full quantity in the Einstein equations and retain terms up to first order in the perturbations. In this chapter we use this setup to study weak gravitational fields on a Minkowski background.

[10]The background solution will be alternatively called the steady-state solution.

2.4 Weak Gravitational Fields on Minkowski Background

2.4.1 Calculating the Einstein Equations

A weak gravitational field on a flat spacetime is represented by a metric that is nearly Minkowski [7] [10][11]:

$$g_{ab}(t,\vec{x}) = \eta_{ab} + h_{ab}(t,\vec{x}) \tag{2.19}$$

where h_{ab} is "small" such that we ignore all terms that are second order and beyond. A weak gravitational field on flat spacetime is a good approximation to gravity far away from black holes and stars. In this section we assume that there are no fields that source gravity. To that effect we set the stress-energy tensor to zero:

$$T_{ab} = 0. \tag{2.20}$$

We checked in the last section that the Minkowski metric is a solution to the Einstein equations in the absence of matter. Now we will substitute the full metric in the Einstein equations and retain terms till linear order in the perturbations. We derive the Christoffel symbols to be:

$$\Gamma^a_{bc} = \frac{\eta^{ad}}{2}(h_{cd,b} + h_{bd,c} - h_{bc,d}). \tag{2.21}$$

[11]We are working in (t,x,y,z) coordinates.

The Ricci tensor is:

$$R_{ab} = h^{c}_{(a,b)c} - \frac{\Box h_{ab}}{2} - \frac{h_{,ab}}{2} \tag{2.22}$$

where h is the trace of the perturbation and $\Box$ is the d'Alembertian or the wave operator which in Cartesian coordinates is:

$$\Box = \partial^a \partial_a = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{2.23}$$

We next calculate the Einstein tensor:

$$G_{ab} = h^{c}_{(a,b)c} - \frac{\Box h_{ab}}{2} - \frac{h_{,ab}}{2} - \frac{\eta_{ab} h^{cd}_{,cd}}{2} + \frac{\eta_{ab} \Box h}{2}. \tag{2.24}$$

To express the Einstein tensor in a simpler algebraic form we define $\bar{h}_{ab}$ [7]:

$$\bar{h}_{ab} = h_{ab} - \frac{\eta_{ab} h}{2}. \tag{2.25}$$

We call $\bar{h}_{ab}$ the trace-reverse of h_{ab} based on the observation that:

$$\bar{h} = -h \tag{2.26}$$

where $\bar{h}$ is the trace of $\bar{h}_{ab}$. Following the re-expression of h_{ab} in terms of its trace-reverse the Einstein tensor in equation (2.24) becomes:

$$G_{ab} = \bar{h}^{c}_{(a,b)c} - \frac{\Box \bar{h}_{ab}}{2} - \frac{\eta_{ab} \bar{h}^{cd}_{,cd}}{2} \tag{2.27}$$

and we note that the algebraic form is much simpler.

2.4.2 Gauge Freedom in the Theory

We are free to choose a coordinate system to represent the metric. If we change coordinates the metric components change. However physical laws are not dependent on coordinates and hence do not change under a coordinate transformation. This coordinate transformation is called a gauge transformation. The freedom to choose a coordinate frame is called gauge freedom and specifying a coordinate system corresponds to fixing a gauge.

We now investigate how the metric perturbations change under a gauge transformation. We specify that the change in coordinates is generated by a small vector $\xi^a(x^b)$ as:

$$x^{a'} = x^a - \xi^a(x^b) \tag{2.28}$$

where $x^{a'}$ is the new coordinate system, x^a is the old one and "small" implies that terms higher than linear order in ξ^a should be ignored. We derive the transformation matrix:

$$\begin{aligned} \lambda_b^{a'} &= \frac{\partial x^{a'}}{\partial x^b} \\ &= \delta_b^a - \xi^a_{,b} \end{aligned} \tag{2.29}$$

where δ_b^a is the Kronecker delta. The matrix for the inverse transformation is given by the relation:

$$\lambda_b^{c'} \lambda_{c'}^a = \delta_b^a \tag{2.30}$$

and we find that to be:

$$\lambda^a_{b'} = \delta^a_b + \xi^a_{,b} \tag{2.31}$$

where we have ignored all terms beyond the first order in ξ^a. We check how the metric changes under this change in coordinates:

$$\begin{aligned} g_{a'b'} &= \lambda^c_{a'}\lambda^d_{b'}g_{cd} \\ g^\star_{ab} &= (\delta^c_a + \xi^c_{,a})(\delta^d_b + \xi^d_{,b})g_{cd} \end{aligned} \tag{2.32}$$

where $\star$ means that the metric is expressed in the new coordinate system. To first order, the coordinate change induces the following transformations:

1. The perturbation h_{ab} transforms as:

$$h^\star_{ab} = h_{ab} + 2\xi_{(a,b)}. \tag{2.33}$$

2. The same for the trace of the perturbation is:

$$h^\star = h + 2\xi^c_{,c}. \tag{2.34}$$

3. Finally, the trace-reversed perturbation transforms as:

$$\bar{h}^\star_{ab} = \bar{h}_{ab} + 2\xi_{(a,b)} - \xi^c_{,c}\eta_{ab}. \tag{2.35}$$

Under the coordinate transformation generated by $\xi^a(x^b)$ we notice that the Einstein tensor stays the same i.e:

$$G^{\star}_{ab} = G_{ab}. \tag{2.36}$$

This implies that the infinitesimal change in coordinate transformation generated by ξ^a did not change the curvature of spacetime. This is what we expected and it served as a check on our gauge-transformed metric perturbation. We conclude that $\bar{h}^{\star}_{ab}$ and $\bar{h}_{ab}$ (and also the pair of $h^{\star}_{ab}$ and h_{ab}) describe the same physical perturbation because they lead to the same curvature.

From here we will proceed forward using two distinct routes. For the first route we will decompose the metric perturbations into different components, fix a gauge and then solve the equations [2][12]. In the second route we will start with equation (2.27), fix a gauge and solve the Einstein equations [7][10][19].

2.4.3 Route 1: Decomposition of Perturbations

The starting point for this route is the realization that the metric perturbation can be broken into components [2]. We start by writing the full line-element:

$$ds^2 = (\eta_{ab} + h_{ab})dx^a dx^b. \tag{2.37}$$

[12]The gauge transformation is generated by the vector ξ^a; consequently specifying ξ^a corresponds to fixing the gauge.

We expand the perturbation part of the line-element as:

$$h_{ab}dx^a dx^b = -h_{00}dt^2 + 2h_{0i}dtdx^i + h_{ij}dx^i dx^j \tag{2.38}$$

and note that h_{00} is a scalar, h_{0i} is a three-vector and h_{ij} is a three-tensor[13]. The three-tensor h_{ij} is a 3×3 matrix that can be further decomposed into a trace and trace-free part:

$$h_{ij} = 2(s_{ij} - \psi e_{ij}) \tag{2.39}$$

where s_{ij} is the traceless part, e_{ij} is the Euclidean metric (and the trace part of the perturbation) and ψ is a function of spacetime. It can be checked that:

$$\psi = -\frac{h}{6} \tag{2.40}$$

where h is the trace of h_{ij}. We denote the different components of the full perturbation h_{ab} as:

$$h_{00} = -2\phi. \tag{2.41}$$

This is one degree of freedom.

$$h_{0i} = w_i. \tag{2.42}$$

[13]In the remaining sections of Chapter 2 and Chapter 3, we denote spacetime indices with the Latin alphabet from the beginning i.e a, b, c.. and spatial indices with the Latin alphabet from the middle i, j, k... This is done to be consistent with the corresponding treatments in the literature mentioned. In Chapters 4 and 5 we will revert back to the abstract index notation [10].

These are three degrees of freedom.

$$h_{ij} = 2(s_{ij} - \psi e_{ij}) \tag{2.43}$$

where s_{ij} has five degrees of freedom and ψ accounts for one. The line element is:

$$ds^2 = -(1+2\phi)dt^2 + 2w_i dt dx^i + [(1-2\psi)e_{ij} + 2s_{ij}]dx^i dx^j. \tag{2.44}$$

We proceed by finding different components of the Einstein tensor which is given in equation (2.24). We state the results for the different components[14]:

$$\begin{aligned}
G_{00} &= 2\nabla^2\psi + s^{kl}_{\ ,kl} \\
G_{0i} &= -\frac{1}{2}\nabla^2 w_i + \frac{1}{2}w^k_{\ ,ik} + 2\dot{\psi}_{,i} + \dot{s}^k_{i,k} \\
G_{ij} &= (e_{ij}\nabla^2 - \partial_i\partial_j)(\phi - \psi) + e_{ij}\dot{w}^k_{\ ,k} - \dot{w}_{(i,j)} + 2e_{ij}\ddot{\psi} - \Box s_{ij} + 2s^k_{(i,j)k} - e_{ij}s^{kl}_{\ ,kl}.
\end{aligned} \tag{2.45}$$

We now check how the different metric functions change under a gauge transformation. Under the gauge transformation generated by ξ^a the metric perturbation transforms as given in equation (2.33). We substitute for the different components of the metric perturbation to get the following trans-

[14]Note that a dot indicates a partial derivative with respect to time

formation properties:

$$\begin{aligned}
\phi^\star &= \phi + 2\dot{\xi}_0 \\
w_i^\star &= w_i + \xi_{0,i} + \dot{\xi}_i \\
\psi^\star &= \psi - \frac{\xi^k_{,k}}{3} \\
s_{ij}^\star &= s_{ij} + \xi_{(i,j)} - \frac{\xi^k_{,k}}{3} e_{ij}.
\end{aligned} \tag{2.46}$$

Now we pick a gauge to solve the Einstein equations. We chose the transverse gauge in which we enforce that the divergences of $w_i^\star$ and $s_{ij}^\star$ vanish i.e:

$$\begin{aligned}
w^{i\star}_{,i} &= 0 \\
s^{i\star}_{j,i} &= 0.
\end{aligned} \tag{2.47}$$

We allow for $w^i_{,i} \neq 0$ and $s^i_{j,i} \neq 0$ in the old coordinate system.

1. For the divergence of w_i to vanish in the new coordinate system we enforce:

   $$\nabla^2 \xi_0 = -w^i_{,i} - \dot{\xi}^i_{,i}. \tag{2.48}$$

   This is one condition on ξ^a and it corresponds to the one restriction $w^i_{,i} = 0$. Therefore there are two degrees of freedom in w^i.

2. The same calculation for s_{ij} gives:

   $$\nabla^2 \xi_j = -2 s^i_{j,i} - \frac{\xi^k_{,jk}}{3}. \tag{2.49}$$

These are three conditions on ξ^a which correspond to the three restrictions $s^i_{j,i} = 0$. The number of degrees of freedom in s_{ij} are reduced to two.

The total number of functions in the metric therefore reduce to six.

In the transverse gauge the Einstein equations simplify[15].

1. The G_{tt} equation becomes:

$$\nabla^2 \psi = 0. \tag{2.50}$$

   The equation above can be solved for ψ to give:

$$\psi = c_1 x + c_0 \tag{2.51}$$

   where c_1 and c_0 are constants of integration. We assume that the functions in the metric are zero at $\pm$ infinity and with these boundary conditions we get:

$$\psi = 0. \tag{2.52}$$

   There are five functions remaining in the metric.

2. The G_{0i} equation is:

$$\nabla^2 w_i = 0 \tag{2.53}$$

   which enforces $w_i = 0$. The number of functions in the metric is three[16].

[15] We no longer use the $\star$ label.

[16] Recall w_i has two functions in the transverse gauge.

3. We now turn to the G_{ij} equations. We first solve the trace of these equations:

$$\nabla^2\phi = 0 \tag{2.54}$$

which indicates that $\phi = 0$. The metric has two degrees of freedom and both of them are in s_{ij}.

4. The trace-free part of the G_{ij} equations is:

$$\Box s_{ij} = 0. \tag{2.55}$$

Therefore in the transverse gauge the remaining two degrees of freedom in the traceless tensor part of the metric perturbation propagate as waves.

We analyze our results for s_{ij}. First we write the solution for equation (2.55):

$$s_{ij} = X_{ij}e^{ik_cx^c} \tag{2.56}$$

where k^c is the wave vector: $k^c = [\omega, \vec{k}]$ with ω representing the frequency of the wave and $\vec{k}$ representing the three wave vector. $e^{ik_cx^c}$ is a plane wave and X_{ij} is its complex constant matrix. We substitute the solution in equation (2.55) to get:

$$k_ck^cs_{ij} = 0. \tag{2.57}$$

This implies that k^a is a null vector. This means that $w^2 = |\vec{k}|^2$. The transverse gauge condition, which is in equation (2.47), imposes that:

$$k_i s^i_j = 0. \tag{2.58}$$

This implies that s_{ij} is transverse to the three wave vector (and hence the motivation to call this gauge the transverse gauge). We now locate where the remaining degrees of freedom lie. Recall that s_{ij} has no time components and that it is symmetric:

$$s_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & s_{11} & s_{12} & s_{13} \\ 0 & s_{12} & s_{22} & s_{23} \\ 0 & s_{13} & s_{23} & s_{33} \end{bmatrix}. \tag{2.59}$$

Following that we invoke that the plane wave which s_{ij} represents is traveling in the z direction. It has the wave vector $k^a = [k, 0, 0, k]^T$. Since s_{ij} is transverse to the wave vector, we can write:

$$s_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & s_{11} & s_{12} & 0 \\ 0 & s_{12} & s_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.60}$$

We lastly enforce that s_{ij} is traceless:

$$s_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & s_{11} & s_{12} & 0 \\ 0 & s_{12} & -s_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.61}$$

These remaining two degrees of freedom propagate as waves: these are gravitational waves.

2.4.4 Route 2: The Harmonic Gauge

In this route we start with equation (2.27). The Einstein equations are:

$$\bar{h}^{c}_{(a,b)c} - \frac{\Box \bar{h}_{ab}}{2} - \frac{\eta_{ab}\bar{h}^{cd}_{,cd}}{2} = 0. \tag{2.62}$$

If we can find a gauge in which:

$$\bar{h}^{b\star}_{a,b} = 0 \tag{2.63}$$

then the Einstein equations in equation (2.62) reduce to a very simple form:

$$\Box \bar{h}^{\star}_{ab} = 0. \tag{2.64}$$

The equation above is significant from a physical standpoint because it indicates that the components of the metric perturbation satisfy the wave equation and hence propagate as waves [7] [10] [19].

We now check if such a gauge exists. We recall the transformation equation for the metric perturbation - i.e equation (2.35) - and take the divergence of both sides to get:

$$\bar{h}^{b\star}_{a,b} = \bar{h}^{b}_{a,b} + \Box\xi_a. \tag{2.65}$$

We allow for $\bar{h}^{b}_{a,b} \neq 0$ in the old coordinate system. Subsequently, if we want a gauge in which $\bar{h}^{b\star}_{a,b} = 0$ then we should solve for ξ^a using the equation:

$$\Box\xi^a = -\bar{h}^{ab}_{,b}. \tag{2.66}$$

This is the wave equation with a source term and we can solve for ξ^a. After solving for ξ^a we get the gauge condition in equation (2.63). This is called the harmonic gauge or the Lorentz gauge and it represents a set of four conditions on the metric perturbations. The Einstein equations simplify to equation (2.64).

We now analyze the equations we have. We start by noting that any plane wave will solve Einstein's equation. We pick a solution:

$$\bar{h}^{\star}_{ab}(x^b) = Z_{ab}e^{ik_cx^c} \tag{2.67}$$

where Z_{ab} is a complex constant matrix. We substitute the solution in the wave equation in equation (2.64) to get:

$$k_c k^c \bar{h}^{\star}_{ab} = 0. \tag{2.68}$$

This implies that k^a is a null vector. It enforces is $w^2 = |\vec{k}|^2$. Upon substituting the solution in the harmonic gauge in equation (2.63) we find:

$$k^b \bar{h}^{\star}_{ab} = 0. \tag{2.69}$$

This set of four conditions restrict the perturbation to be orthogonal (or transverse) to the four wave vector.

We next check if there is more gauge-freedom in the theory. We start by noting that the Lorentz gauge is equally possible if we add to ξ^a another vector γ^a that satisfies:

$$\Box \gamma^a = 0. \tag{2.70}$$

γ^a also satisfies the wave equation and we can specify a solution as:

$$\gamma^a = Y^a e^{ik_c x^c} \tag{2.71}$$

where Y^a is a constant vector. That we can add ξ^a and γ^a indicates that the Lorentz gauge is actually a class of gauges: that it represents not one, but several gauges that satisfy equation (2.66) for ξ^a. It suggests that we have more gauge freedom which we can use to put further restrictions on the perturbation. We write the transformation equation for the metric perturbation generated by γ^a[17]:

$$\bar{h}^{\star}_{ab} = \bar{h}_{ab} + 2\gamma_{(a,b)} - \gamma^{c}_{,c}\eta_{ab}. \tag{2.72}$$

Because both $\bar{h}_{ab}$ and γ^a satisfy the wave equation, we next rewrite the above in terms of solutions of the wave equation given in equations (2.67) and (2.71):

$$Z^{\star}_{ab} = Z_{ab} + 2ik_{(b}Y_{a)} - iY^c k_c \eta_{ab}. \tag{2.73}$$

We first look for a condition on γ^a that renders traceless, the metric perturbation in the new coordinate system. This enforces the condition:

$$Y^a k_a = -\frac{iZ}{2} \tag{2.74}$$

where Z is the trace of Z_{ab}[18]. This is one condition on Y^a and subsequently on γ^a. We still have three more conditions we can enforce. We now enforce that in the new coordinates, the metric perturbation should be orthogonal to the timelike four velocity U^a of some observer. Equation (2.73) becomes:

$$0 = Z_{ab}U^a + 2iY_{(a}k_{b)}U^a - iY^c k_c U_b. \tag{2.75}$$

It seems as if the equation above is indicative of four conditions (one for

[17]This is the same as equation (2.35) but with ξ^a replaced by γ^a

[18]We can also impose other conditions. For example we can set $Z^{\star}_{22} = 0$. We impose tracelessness because we saw in Route 1 that the propagating degrees of freedom were in the traceless part of the metric perturbation

each free index). We note that that is not necessarily the case: if we further project using k^b, we get zero. This implies we are in the three-dimensional plane orthogonal to k^b and one condition from equation (2.75) is redundant. We have now used up all our gauge freedom and the conditions on the metric perturbation are[19]:

$$\begin{aligned} \bar{h} &= 0 \\ U^a \bar{h}_{ab} &= 0. \end{aligned} \tag{2.76}$$

We additionally note that in this gauge because the metric perturbation is traceless:

$$\bar{h}_{ab} = h_{ab}. \tag{2.77}$$

Now we will collect our results and analyze them. The total number of degrees of freedom in the symmetric metric perturbation is ten and they satisfy the wave equation (2.64). We write the symmetric metric perturbation in matrix form as:

$$h_{ab} = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{03} \\ - & h_{11} & h_{12} & h_{13} \\ - & - & h_{22} & h_{23} \\ - & - & - & h_{33} \end{bmatrix}. \tag{2.78}$$

The Lorentz gauge imposes that the metric perturbation is transverse to the

[19]We will not be using the $\star$ superscript anymore.

four wave vector.

$$h_{ab}k^b = 0. \tag{2.79}$$

We choose the wave to travel in the z-direction i.e:

$$k^a = [k, 0, 0, k]^T. \tag{2.80}$$

The metric perturbation becomes:

$$h_{ab} = \begin{bmatrix} h_{00} & h_{01} & h_{02} & -h_{00} \\ - & h_{11} & h_{12} & -h_{01} \\ - & - & h_{22} & -h_{02} \\ - & - & - & -h_{00} \end{bmatrix}. \tag{2.81}$$

We then impose that the metric perturbation is orthogonal to a timelike four-velocity:

$$h_{ab}U^b = 0. \tag{2.82}$$

These are three restrictions on the metric perturbation. We choose a frame in which the four-velocity vector is:

$$U^a = [1, 0, 0, 0]^T. \tag{2.83}$$

This imposes the metric perturbation to be:

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ - & h_{11} & h_{12} & 0 \\ - & - & h_{22} & 0 \\ - & - & - & 0 \end{bmatrix}. \tag{2.84}$$

We finally are left with the traceless condition:

$$h = 0. \tag{2.85}$$

This is the final condition on the metric perturbation and it enforces it to be:

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{12} & -h_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.86}$$

We are left with two degrees of freedom in the metric perturbation. These satisfy the wave equation. We should additionally note that they are not sourced by any matter. These are gravitational waves.

2.4.5 Comparing Routes

Each route had the same two starting points:

1. An expression for the Einstein tensor to linear order in the perturba-

tion.

2. The understanding that there exists a gauge freedom which can be used to restrict the form of the metric perturbation without changing the physics.

The different methods of analyzing weak gravitational fields led to the same result of propagating degrees of freedom in the traceless, transverse part of the metric perturbation. We call these gravitational waves. It is worth noting that the perturbations can be expressed as:

$$h_{ij} = h_{11} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + h_{12} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.87}$$

If we define two matrices A and B to be orthogonal if $Tr(AB) = Tr(BA) = 0$, then the two matrices in the equation above are orthogonal. Consequently we have expressed our metric perturbation in terms of an orthogonal basis which is independent of position and time. The coefficients of the basis matrices are scalar functions of spacetime i.e they generate a number for each point in spacetime. The gravitational wave equations are:

$$\begin{aligned} \Box h_{11} &= 0 \\ \Box h_{12} &= 0. \end{aligned} \tag{2.88}$$

This ends the treatment of the weak field case. The gravitational wave equations will be further analyzed in later sections.

2.5 Summary

In this chapter we reviewed the Einstein equations that relate the geometry of spacetime to matter. We found that for a system without matter the simplest solution to the Einstein equations is the Minkowski metric. Following that we used perturbation theory in GR to study weak fields on a Minkowski background.

We used two routes. For the first route we decomposed the metric perturbations into scalar, vector and tensor components. The tensor - on account of being a 3×3 matrix - was further decomposed into a trace and traceless part. We next derived the Einstein equations for the metric perturbations and studied them in the transverse gauge. This gauge enforced the vector and traceless tensor perturbations to be divergenceless in space. This accounted for four conditions on the metric. We solved the Einstein equations in the transverse, traceless gauge and noted that the scalar and vector components of the perturbation vanish. The remaining two degrees of freedom were in the traceless, transverse tensor components and they satisfied the wave equation. Using this route we concluded that the degrees of freedom that correspond to gravitational waves were in the traceless, transverse tensor components of the metric perturbations. These two degrees of freedom that propagated as waves were called gravitational waves.

For the second route we employed the harmonic gauge. We found that the metric perturbations satisfy the wave equation. We checked that the harmonic gauge allows us to put eight restrictions on our metric perturbations. The remaining two degrees of freedom came from the traceless part of the metric perturbations which was transverse to the four wave vector and also to a time-like four velocity. This ended the treatment of the weak field case using the harmonic gauge.

We also saw that these degrees of freedom can be expressed in terms of scalar coefficients in an orthogonal matrix basis.

Chapter 3

Cosmological Perturbation Theory

This is a review of cosmological perturbation theory [6] [13]. In Section 3.1 we discuss the simplest possible mathematical representation of the cosmological spacetime and matter. We then study perturbation theory on this spacetime and matter.

3.1 Cosmology

Cosmology is the study of the universe at scales of millions of light-years. The Einstein field equations - as it will be seen shortly - simplify greatly when we study the universe at such large scales. The subsequent insights provided are very rich because through them one may chart the history of the universe and predict its fate. In this section we will study the simplest possible universe through GR.

Einstein's equations require the metric and stress-energy tensors. For cosmology the forms of these inputs are grounded in philosophical prejudices and observations [10]. These presuppositions about the universe date back to Copernicus in the 16th century. The Copernican Principle states that we are not in a privileged position in the universe. By extension if we were in a different part of the universe our observations would be unchanged. The Copernican Principle, which is also known as the Principle of Ordinariness, therefore removes from our observations a dependence on position. This is referred to as homogeneity. It was also presupposed that there are no preferred directions in space: that observations along $\hat{\imath}$ should not be different from the observations along $\hat{\jmath}$ or $\hat{k}$[1]. This is called isotropy. These philosophical biases are confirmed by observations. For example we find that the universe is filled with a background thermal radiation of 2.7 Kelvin. This is called the Cosmic Microwave Background (CMB) and its temperature is measured to be the same in every direction[2]. The CMB is a strong indicator of isotropy.

We now derive the simplest metric for the cosmological spacetime. We choose a coordinate system such that there is no mixing of time and spa-

[1]At this point it is imperative to reiterate that this discussion is valid on scales of millions of light years. On the scale of the solar system - for example - these prejudices do not work.

[2]Fluctuations to the CMB are no more than ± 0.001 Kelvin. This point will become important when we discuss perturbations

tial components. This removes all $dtdx^i$ terms from the metric. We next presuppose spatial flatness which means that distances in space are calculated using Pythagoras' Theorem. By extension the line element in (t, x, y, z) coordinates is:

$$ds^2 = -dt^2 + a_x^2(x^b)dx^2 + a_y^2(x^b)dy^2 + a_z^2(x^b)dz^2 \tag{3.1}$$

where $a_i(t, x, y, z)$ represents the expansion in the $\hat{i}$ direction and is called the scale factor. We recall that the Copernican Principle implies that the expansion of the universe does not depend on position and is homogeneous. The line-element for cosmology gets modified to:

$$ds^2 = -dt^2 + a_x^2(t)dx^2 + a_y^2(t)dy^2 + a_z^2(t)dz^2. \tag{3.2}$$

Isotropy further adds to the simplicity already imposed. Equal expansions in all directions enforces the line-element of cosmology to:

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)dx^2 + a^2(t)dy^2 + a^2(t)dz^2 \\ ds^2 &= -dt^2 + a^2(t)e_{ij}dx^i dx^j \end{aligned} \tag{3.3}$$

where e_{ij} is the Euclidean metric. The cosmological metric in matrix form is:

$$g_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix}. \tag{3.4}$$

As mentioned before observations of matter in the large scale universe also indicate homogeneity and isotropy. These qualities should be reflected by the stress-energy tensor that represents this matter. For example the energy-representing quantities should not be functions of position. If there exists a pressure then it should be the same in all directions. It should also be noted that any 3-velocity of the matter distribution violates isotropy; hence the matter distribution should be assumed to be at rest which enforces the four-velocity dual vector to be:

$$u_a = [-1, 0, 0, 0]. \tag{3.5}$$

We use the perfect fluid to model the matter distribution in the homogeneous and isotropic universe. The stress-energy tensor for a perfect fluid was specified in equation (2.3). In matrix form it is:

$$T_a^b = \begin{bmatrix} -\rho^{(0)} & 0 & 0 & 0 \\ 0 & p^{(0)} & 0 & 0 \\ 0 & 0 & p^{(0)} & 0 \\ 0 & 0 & 0 & p^{(0)} \end{bmatrix} \tag{3.6}$$

in which every variable is only a function of time[3]. The superscripts $^{(0)}$ indicate that this quantity represents the homogeneous and isotropic universe which we alternatively call the Friedmann–Lemaitre–Robertson–Walker (FLRW) universe.

We now have the inputs to use GR to investigate our universe.

The Einstein tensor takes the form:

$$G^a_b = \begin{bmatrix} -3H^2 & 0 & 0 & 0 \\ 0 & -H^2 - \frac{2\ddot{a}}{a} & 0 & 0 \\ 0 & 0 & -H^2 - \frac{2\ddot{a}}{a} & 0 \\ 0 & 0 & 0 & -H^2 - \frac{2\ddot{a}}{a} \end{bmatrix} \tag{3.7}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble factor. Before proceeding we note that the simple forms of both - the Einstein and stress-energy - tensors arise because of the symmetries imposed on the background cosmological spacetime and matter-energy density. The identical elements along the diagonal for the spatial components are manifestations of isotropy. The variables being position-independent indicates homogeneity. The Einstein field equations become simpler because of the symmetries present in the FLRW universe.

We now derive the different components of equation (2.1). The $(0,0)$

[3]We introduce the stress-energy tensor with one index up because factors of a get canceled and this makes its form simpler. We will extend this practice to the Einstein tensor when we compute the Einstein equations.

equation is:

$$\begin{aligned} G^0_0 &= 8\pi G T^0_0 \\ H^2 &= \frac{8\pi G}{3}\rho^{(0)}. \end{aligned} \tag{3.8}$$

The spatial equations are:

$$\begin{aligned} G^i_j &= 8\pi G T^i_j \\ H^2 + \frac{2\ddot{a}}{a} &= -8\pi G p^{(0)}. \end{aligned} \tag{3.9}$$

One may re-express both equations as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p^{(0)} + \rho^{(0)}). \tag{3.10}$$

We next solve for equation (2.6):

$$\begin{aligned} &T^a_{b;a} = 0 \\ &\dot{\rho}^{(0)} = -3H(\rho^{(0)} + p^0). \end{aligned} \tag{3.11}$$

3.1.1 Example: Solution in a Dust-Dominated Universe

The equations above can be used to study different epochs of the universe. In the current epoch of the universe the pressure in the stress-energy tensor is zero. This indicates that the interactions between matter at the largest of scales is negligible. Matter that behaves this way is called "dust" and it is represented by the stress-energy tensor:

$$T_{ab} = \rho u_a u_b. \tag{3.12}$$

In the current - dust dominated - epoch, the spatial equations (3.9) are[4]:

$$\ddot{a} = -\frac{\dot{a}^2}{2a}. \tag{3.13}$$

We also note that equation (3.11) becomes an ordinary differential equation, which can be solved to give:

$$\rho = \frac{\rho_0}{a^3} \tag{3.14}$$

where ρ_0 is a constant[5]. This implies that the density of the matter field is inversely proportional to the volume of the universe. Therefore as the universe expands it becomes dilute. We now substitute these results in equation (3.10) and solve for the scale factor.

$$\begin{aligned} a\dot{a}^2 &= \frac{8\pi G}{3}\rho_0 \\ \sqrt{a}da &= \sqrt{\frac{8\pi G}{3}\rho_0}dt. \end{aligned} \tag{3.15}$$

[4]This simplification will be used extensively when we calculate the perturbed equations of motion.

[5]We have dropped the superscript $^{(0)}$ for this example. It should be understood that we are dealing with the homogeneous and isotropic universe.

The equation above can be solved to give:

$$a(t) = (6\pi G\rho_0)^{\frac{1}{3}}(t - t_0)^{\frac{2}{3}} \tag{3.16}$$

where t_0 is the initial time for which the scale factor is zero.

This ends our treatment of the FLRW universe. The dynamics found are the background solutions on which we will study the perturbations.

3.2 Introducing Perturbations

In this section we introduce perturbations to the homogeneous and isotropic cosmological metric and stress-energy tensor.

We start with the metric. Recall the full line element we had written down in Minkowski spacetime in equation (2.44). We note that in cosmology every space component gets multiplied by the scale factor to account for the expansion. Therefore the full line element of cosmology can be written as:

$$ds^2 = -(1 + 2\phi)dt^2 + 2a(t)w_i dt dx^i + a^2[(1 - 2\psi)e_{ij} + 2s_{ij}]dx^i dx^j. \tag{3.17}$$

We recall that ϕ and ψ are scalars, w_i is a three-vector and s_{ij} is a traceless 3×3 matrix.

We now introduce perturbations in the stress-energy tensor:

$$T^a_b = (\rho + p)u^a u_b + p\delta^a_b. \tag{3.18}$$

Note that ρ, p and u_a are perturbed. We

1. Break isotropy by equipping the fluid with a three-velocity v^i. Following that we make all perturbations functions of space to break homogeneity. Under these requirements:

$$u_a(t, x^i) = [-1 - \frac{g_{00}^{(1)}}{2}(t, x^i), a(t)v_i^{(1)}(t, x^j)]$$
$$\rho(t, x^i) = \rho^{(0)}(t) + \rho^{(1)}(t, x^i) \tag{3.19}$$
$$p(t, x^i) = p^{(0)}(t) + p^{(1)}(t, x^i)$$

where the superscript $^{(1)}$ represents a perturbation. Note that the four-velocity is a unit time-like vector. This enforces the unit four-velocity vector to be:

$$u^a = [1 - \frac{g_{00}^{(1)}}{2}(t, x^i), \frac{e^{ik}(v_k^{(1)} - g_{0k}^{(1)})}{a}]^T. \tag{3.20}$$

We have scalar perturbations and one three-vector perturbation. To keep the treatment general we add a three-tensor perturbation $\Sigma_j^{i(1)}(t, x^k)$. This is referred to as anisotropic stress.

2. The different elements of the full T^a_b are:

$$\begin{aligned} T^0_0 &= -\rho \\ T^0_i &= (p^{(0)} + \rho^{(0)}) a v^{(1)}_i \\ T^i_0 &= -(p^{(0)} + \rho^{(0)}) \frac{e^{ik}(v^{(1)}_k - g^{(1)}_{0k})}{a} \\ T^i_j &= \delta^i_j p + \frac{e^{ik}}{a^2} \Sigma^{(1)}_{kj}. \end{aligned} \tag{3.21}$$

3. To get the stress-energy tensor in a simpler algebraic form we prescribe that:

$$q^{i(1)} = (p^{(0)} + \rho^{(0)}) a v^{i(1)}. \tag{3.22}$$

Under this definition the full stress energy tensor is[6]:

$$\begin{aligned} T^0_0 &= -\rho \\ T^0_i &= q_i \\ T^i_0 &= \frac{e^{ik}}{a^2}[-q_k + a(p^{(0)} + \rho^{(0)}) g^{(1)}_{0k}] \\ T^i_j &= \delta^i_j p + \frac{e^{ik}}{a^2} \Sigma_{jk}. \end{aligned} \tag{3.23}$$

In the next section we will discuss the decomposition of these perturbations.

[6]It is understood that v, q and Σ only occur as perturbations. Therefore from now on the label $^{(1)}$ will not be used

3.3 Decomposition Based on Rotations

Vectors and tensors can be decomposed and expressed in terms of basis. The decomposition gives meaning to the components of these geometrical objects. For example one may decompose a vector in the Cartesian basis and then study the components in the x-direction. In this section a basis will be constructed based on how the basis elements respond to rotations in Fourier space by an angle θ. We will use this basis to decompose our perturbations.

3.3.1 Fourier Transform

We work in Fourier space. This necessitates taking a Fourier transform of all functions of space. If one starts off with an object that is a function of space variables then the Fourier transform allows one to view that object in terms of of plane waves with some coefficients. Assume one has the three-tensor $M^{ij}(t, x^k)$; this is expressed in Fourier space as:

$$M^{ij}(t, x^l) = \int d^3k e^{i\vec{k}.\vec{x}} \tilde{M}^{ij}(t, k^l) \tag{3.24}$$

where $\tilde{M}^{ij}$ is the Fourier transform, k^i is the Fourier mode or the three wave vector and $e^{i\vec{k}.\vec{x}}$ is the plane wave[7]. If there is a spatial derivative acting on equation (3.24) then on the right hand side it will pull down a factor of ik_j

[7]Note that the plane wave has no time-dependence. The time-dependence is in the coefficient.

and time derivative will act on the coefficient of the plane wave.

As an exercise we express the gravitational wave equations for weak fields on flat spacetime - equation (2.88) - in Fourier space. We define:

$$h_{11}(t, x^l) = \int d^3k e^{i\vec{k}.\vec{x}} \tilde{h}_{11}(t, k^l). \tag{3.25}$$

We substitute for h_{11} and find that the gravitational wave equation becomes:

$$\ddot{\tilde{h}}_{11} = -k^2 \tilde{h}_{11} \tag{3.26}$$

where k^2 represents the square of the magnitude of the three wave vector.

3.3.2 Constructing the Basis

To construct the basis mentioned previously start with a Fourier mode k^i. Perpendicular to that one has a two-dimensional plane. Pick e_1^i and e_2^i to be orthonormal vectors spanning that plane. One may - for convenience - relate $[e_1^i, e_2^i, \hat{k}^i]$ with $[\hat{i}, \hat{j}, \hat{k}]$. Consider rotations in that plane by angle θ, an operation performed by J_θ:

$$J_\theta = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 \\ -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.27}$$

One may then write the following eigenvalue equations:

$$J_\theta e^i_\pm = e^{\pm i\theta} e^i_\pm \tag{3.28}$$

and

$$J_\theta e^i_3 = e^i_3 \tag{3.29}$$

where

$$e^i_\pm = \frac{e^i_1 \pm i e^i_2}{\sqrt{2}} \tag{3.30}$$

and

$$e^i_3 = \hat{k}^i. \tag{3.31}$$

A vector in Fourier space may be decomposed using the orthonormal basis $e^i_\pm$ (which have eigenvalues $e^{\pm i\theta}$) and e^i_3 (which has an eigenvalue of 1). For decomposition of three-tensors one may construct an orthonormal basis using outer products of these three unit vectors. The rotation operator will act on each basis vector separately as such:

$$\begin{aligned} J_\theta(e^j_+ \times e^k_+) &= (J_\theta e^j_+) \times (J_\theta e^k_+) \\ &= (e^{i\theta} e^j_+) \times (e^{i\theta} e^k_+) \\ &= e^{2i\theta}(e^j_+ \times e^k_+). \end{aligned} \tag{3.32}$$

Possible basis elements are $e^i_- \times e^j_-$ (with eigenvalue $e^{-2i\theta}$), $e^i_+ \times e^j_-$ (with eigenvalue 1) and $e^i_- \times e^j_3$ (with eigenvalue $e^{-i\theta}$). If we denote the eigenvalue by $e^{im\theta}$ - where m is an integer - then a scalar can only correspond to $m = 0$, the basis elements for a three-vector can take values between $m = -1$ to $m = 1$ and for a basis element of a three-tensor m can go from -2 to 2.

Based on the eigenvalue equation we define a basis element to be a:

1. Helicity scalar if $m = 0$,

2. Helicity vector if $m = \pm 1$,

3. Helicity tensor if $m = \pm 2$.

We can also conclude that:

1. A scalar in Fourier space may only correspond to a helicity scalar.

2. For a three-vector in Fourier space, $\tilde{V}^i$:

$$\tilde{V}^i = \tilde{v}_+ e^i_+ + \tilde{v}_- e^i_- + \tilde{v}_3 e^i_3. \tag{3.33}$$

   Therefore a three-vector has helicity vector components (with $m = \pm 1$) and a helicity scalar part. As seen in equation (3.33) the helicity vector components are transverse (orthogonal) to the Fourier mode.

3. Lastly three-tensors in Fourier space may be decomposed in all three helicities[8].

   (a) The helicity scalar components are formed by linear combinations of $e^i_+ e^j_-$, $e^i_- e^j_+$ and $e^i_3 e^j_3$. For example one of the basis elements is

[8]For examples please consult chapter 5.

e^{ij}:

$$e^{ij} = 2e^{i}_{(+}e^{j}_{-)} + e^{i}_{3}e^{j}_{3}. \quad (3.34)$$

(b) Helicity vector components are formed by linear combinations of $e^{(i}_{+}e^{j)}_{3}$ and $e^{(i}_{-}e^{j)}_{3}$. We define a helicity vector M^{ij} to be transverse if it satisfies $M^{ij}e^{3}_{i}e^{3}_{j} = 0$.

(c) The helicity tensor basis are formed by linear combinations of $e^{i}_{+}e^{j}_{+}$ and $e^{i}_{-}e^{j}_{-}$. Therefore for a helicity tensor M^{ij} to be transverse, it need only satisfy $M^{ij}e^{3}_{i} = 0$.

As an example a three-tensor in Fourier space can be written as:

$$\tilde{R}^{ij} = (e^{i}_{3}e^{j}_{3} + be^{ij})R_S + e^{i}_{+}e^{j}_{3}R_V + e^{i}_{-}e^{j}_{-}R_T \quad (3.35)$$

where b is a constant and the subscripts S, V and T represent the coefficients for the helicity scalar, helicity vector and helicity tensor basis elements respectively.

3.3.3 Interpreting the Basis

We now check what our findings from the previous subsection mean if we do not take a Fourier transform of our functions.

A three-vector from equation (3.33) can be written as:

$$V_i = v_{1i} + v_{2i} + v_{,i} \quad (3.36)$$

where v is a helicity scalar[9] and v_{1i} and v_{2i} are helicity vectors that are divergenceless:

$$v^i_{,i} = 0. \tag{3.37}$$

The equation above corresponds in Fourier space to the helicity vector being transverse.

A three-tensor can be expressed as:

$$T_{ij} = t_{ij} + 2t_{(i,j)} + [\partial_i \partial_j - c e_{ij} \nabla^2] t \tag{3.38}$$

where

1. t_{ij} is the helicity tensor. It should be divergenceless i.e[10]:

$$t^i_{j,i} = 0. \tag{3.39}$$

   Note that if T_{ij} is traceless, then $t^i_i = 0$.

2. t_i is a helicity vector. If T_{ij} is trace-free then the helicity vector should be divergenceless i.e: $t^i_{,i} = 0$[11].

3. t is a helicity scalar.

4. c is a constant. If T_{ij} is traceless then $c = -\frac{1}{3}$.

[9] Recall that a partial derivative corresponds in Fourier space to multiplication by k_i.

[10] This corresponds in Fourier space to t_{ij} being transverse to the Fourier mode.

[11] In Fourier space, this would mean that the helicity vector is transverse to the Fourier mode.

In this section we generalized the interpretation of helicity beyond the context of Fourier space. We will use the results from this section to decompose our metric and stress-energy tensor perturbations[12].

3.4 Decomposing Perturbations Based on Helicities

The full line element of cosmology is:

$$ds^2 = -(1+2\phi)dt^2 + 2a(t)w_i dt dx^i + a^2[(1-2\psi)e_{ij} + 2s_{ij}]dx^i dx^j. \quad (3.40)$$

We recall that ϕ and ψ are scalars, w_i is a three-vector and s_{ij} is a traceless 3×3 matrix. We find that:

1. ϕ and ψ correspond only to a helicity scalars. We have two degrees of freedom here.

2. w_i - a three-vector - has a helicity scalar (which we denote by B) and helicity vector components (which we call S_i). We set:

$$2w_i = B_{,i} + S_i. \quad (3.41)$$

   We note - from the discussion in the previous chapter - that S_i is divergenceless. Therefore we have one degree of freedom in B and two

[12] We can also use this decomposition in Section 2.4.3.

in S_i.

3. s_{ij} - a traceless three-tensor - has a helicity scalar (E), helicity vector (F_i) and helicity tensor (h_{ij}) components. Keeping in line with the prescriptions given in the relevant literature we impose that:

$$2s_{ij} = 2(\partial_i \partial_j - \frac{e_{ij}}{3}\nabla^2)E + 2F_{(i,j)} + h_{ij}. \qquad (3.42)$$

We note that s_{ij} being traceless implies that $F^i_{,i} = 0$ and $h^i_i = 0$. This is one condition each on F_i and h_{ij}. Keeping in line with the discussion in the previous section we conclude that h_{ij} is divergenceless:

$$h^i_{j,i} = 0. \qquad (3.43)$$

These are three conditions on h_{ij}. In counting the number of independent functions we note that E accounts for one, F_i accounts for two and that h_{ij} also accounts for two.

We finally point out that is customary to absorb the $\nabla^2 E$ term in ψ.

This decomposition is extended to the perturbations to the stress-energy tensor. The perturbation part of this tensor is:

$$
\begin{aligned}
T_0^{0(1)} &= -\rho^{(1)} \\
T_i^{0(1)} &= q_i \\
T_0^{i(1)} &= \frac{e^{ik}}{a^2}[-q_k + a(p^{(0)} + \rho^{(0)})g_{0k}^{(1)}] \\
T_j^{i(1)} &= \delta_j^i p^{(1)} + \frac{e^{ik}}{a^2}\Sigma_{jk}.
\end{aligned}
\tag{3.44}
$$

We find that:

1. $\rho^{(1)}$ and $p^{(1)}$ are helicity scalars.

2. q_k can be decomposed in terms of a helicity scalar and helicity vectors.

3. Σ_{jk} can be decomposed in terms of all three helicities mentioned[13].

Now we proceed to find the Einstein equations for each category of perturbations.

3.5 Scalar Perturbation Equations

The helicity scalar perturbations in the line element are:

$$ds^2 = -(1+2\phi)dt^2 + aB_{,i}dtdx^i + a^2[(1-2\psi)e_{ij} + 2E_{,ij}]dx^i dx^j. \tag{3.45}$$

[13]The different components of q_k and Σ_{ij} will be specified later.

The corresponding perturbations in stress-energy tensor are:

$$
\begin{aligned}
T_0^{0(1)} &= -\rho^{(1)} \\
T_i^{0(1)} &= q_{,i} \\
T_0^{i(1)} &= \frac{e^{ik}}{a^2}[-q_{,k} + a(p^{(0)} + \rho^{(0)})B_{,k}] \\
T_j^{i(1)} &= \delta_j^i p^{(1)} + \frac{e^{ik}}{a^2}\Sigma_{,jk}.
\end{aligned}
\tag{3.46}
$$

The spatial Fourier transform of the Einstein equations at linear order are as follows:

$$
\begin{gathered}
\tilde{G}_0^0 = 8\pi G \tilde{T}_0^0 \\
3H(\dot{\tilde{\psi}} + H\tilde{\phi}) + \frac{k^2}{a^2}[\tilde{\psi} + H(a^2\dot{\tilde{E}} - a\tilde{B})] = -4\pi G\tilde{\rho}^{(1)}
\end{gathered}
\tag{3.47}
$$

$$
\begin{gathered}
\tilde{G}_i^0 = 8\pi G \tilde{T}_i^0 \\
\dot{\tilde{\psi}} + H\tilde{\phi} = -4\pi G\tilde{q}.
\end{gathered}
\tag{3.48}
$$

The (i, j) equations for $i \neq j$ are:

$$
\begin{gathered}
\tilde{G}_j^i = 8\pi G \tilde{T}_j^i \\
\ddot{\tilde{E}} + 3H\dot{\tilde{E}} - \frac{\dot{\tilde{B}}}{a} - \frac{2H\tilde{B}}{a} + \frac{\tilde{\psi} - \tilde{\phi}}{a} = \frac{8\pi G\tilde{\Sigma}}{a^2}.
\end{gathered}
\tag{3.49}
$$

The trace part of the spatial set of equations is:

$$\tilde{G}^i_i = 8\pi G \tilde{T}^i_i \tag{3.50}$$

$$\begin{aligned} \ddot{\tilde{\psi}} + 3H\dot{\tilde{\psi}} + H\dot{\tilde{\phi}} + \frac{|k|^2}{3}\left[\ddot{\tilde{E}} + 3H\dot{\tilde{E}} - \frac{\dot{\tilde{B}}}{a} - \frac{2H\tilde{B}}{a} + \frac{(\tilde{\psi} - \tilde{\phi})}{a^2}\right] = \\ 4\pi G\tilde{p}^{(1)} - 4\pi G\frac{|k|^2}{3a^2}\tilde{\Sigma}. \end{aligned} \tag{3.51}$$

We note that the equation above can be simplified using equation (3.49). The simplified version of equation (3.51) is:

$$\begin{aligned} \tilde{G}^i_i &= 8\pi G \tilde{T}^i_i \\ \ddot{\tilde{\psi}} + 3H\dot{\tilde{\psi}} + H\dot{\tilde{\phi}} &= 4\pi G\left[\tilde{p}^{(1)} - \frac{|k|^2}{a^2}\tilde{\Sigma}\right]. \end{aligned} \tag{3.52}$$

The spatial Fourier transform of the remaining equations are:

$$\begin{aligned} \tilde{T}^a_{0;a} \equiv \dot{\tilde{\rho}}^{(1)} + 3H(\tilde{\rho}^{(1)} + \tilde{p}^{(1)}) = \\ \frac{|k|^2}{a^2}\tilde{q} + (\rho^{(0)} + p^{(0)})\left[3\dot{\tilde{\psi}} + |k|^2\left(\dot{\tilde{E}} - \frac{\tilde{B}}{a}\right)\right] + \frac{H|k|^2}{a^2}\tilde{\Sigma} \end{aligned} \tag{3.53}$$

$$\tilde{T}^a_{i;a} \equiv \dot{\tilde{q}}^{(1)} + 3H\tilde{q}^{(1)} = \frac{|k|^2}{a^2}\tilde{\Sigma} - (\rho^{(0)} + p^{(0)})\tilde{\phi} - \tilde{p}^{(1)}. \tag{3.54}$$

Therefore the scalar modes exhibit non-trivial dynamics. However, these equations are not analyzed further. We proceed now to study helicity vector perturbations.

3.6 Vector Perturbation Equations

The helicity vector perturbations in the line element are:

$$ds^2 = -dt^2 + aS_i dt dx^i + a^2[e_{ij} + 2F_{(i,j)}]dx^i dx^j \tag{3.55}$$

with the restrictions that the perturbations do not diverge:

$$\begin{aligned} S^i_{,i} &= 0 \\ F^i_{,i} &= 0. \end{aligned} \tag{3.56}$$

The corresponding perturbations in the stress-energy tensor are:

$$\begin{aligned} T_0^{0(1)} &= 0 \\ T_i^{0(1)} &= q_i \\ T_0^{i(1)} &= \frac{e^{ik}}{a^2}[-q_k + a(p^{(0)} + \rho^{(0)})S_k] \\ T_j^{i(1)} &= \frac{e^{ik}}{a^2}\Sigma_{(j,k)}. \end{aligned} \tag{3.57}$$

The $(0,0)$ component of the Einstein equation is identically zero:

$$G_0^0 = 8\pi G T_0^0 = 0. \tag{3.58}$$

The spatial Fourier transform of the $(0, i)$ component is:

$$\begin{gathered} \tilde{G}^0_i = 8\pi G \tilde{T}^0_i \\ |k|^2 \left(\dot{\tilde{F}}_i - \frac{\tilde{S}_i}{a} \right) = 16\pi G \tilde{q}_i. \end{gathered} \tag{3.59}$$

Under the redefinition:

$$v_i = \dot{F}_i - \frac{S_i}{a} \tag{3.60}$$

equation (3.59) becomes:

$$|k|^2 \tilde{v}_i = 16\pi G \tilde{q}_i. \tag{3.61}$$

The spatial Fourier transform of the (i, j) equation is:

$$\begin{gathered} \tilde{G}^i_j = 8\pi G \tilde{T}^i_j \\ \dot{\tilde{v}}_{(i} k_{j)} + 3H \tilde{v}_{(i} k_{j)} = \frac{4\pi G \tilde{\Sigma}_{(i} k_{j)}}{a^2}. \end{gathered} \tag{3.62}$$

If one looks at the (i, i) part of these equations one may peel off the Fourier mode dual vector to get:

$$\dot{\tilde{v}}_i + 3H \tilde{v}_i = \frac{4\pi G \tilde{\Sigma}_i}{a^2}. \tag{3.63}$$

In the absence of the anisotropic stress the resultant ordinary differential equation has the solution:

$$v_i \propto \frac{1}{a^3}. \tag{3.64}$$

Therefore the metric perturbation decays with an expanding universe. In the absence of anisotropic stress the spatial Fourier transform of the remaining equation is:

$$\tilde{T}^{a}_{i;a} \equiv \dot{\tilde{q}}_i + 3H\tilde{q}_i = 0. \tag{3.65}$$

Therefore q_i follows a similar fate to that of the metric perturbations; it also decays with the expansion of the universe. This is further confirmed by equation (3.61); the system therefore is consistent.

The vector modes - in this case - may thus be ignored. We next study the tensor modes.

3.7 Tensor Perturbation Equations

The helicity tensor perturbations in the line element are:

$$ds^2 = -dt^2 + a^2(e_{ij} + h_{ij})dx^i dx^j \tag{3.66}$$

where the symmetric h_{ij} is transverse and traceless:

$$\begin{aligned} h^i_{j,i} &= 0 \\ h^i_i &= 0. \end{aligned} \tag{3.67}$$

We recall that previously when studying weak gravitational fields we found the gravitational wave equation through the transverse, traceless metric perturbation. Therefore we expect that the equations of motion we get from the helicity tensor modes represent gravitational waves in the universe. We also note that there are only two degrees of freedom left.

To a reasonable approximation the helicity tensor perturbation is sourced by matter that is negligible; one may therefore set the stress-energy tensor to zero.

The spatial Fourier transform of the Einstein field equation gives:

$$\ddot{\tilde{h}}_{ij} = -\frac{|k|^2}{a^2}\tilde{h}_{ij} - 3H\dot{\tilde{h}}_{ij}. \tag{3.68}$$

Recall that a helicity tensor has $m = \pm 2$. Thus one may re-express $\tilde{h}_{ij}$ as:

$$\tilde{h}_{ij}(t, k^i) = h_+(t, k^i)H^+_{ij} + h_-(t, k^i)H^-_{ij} \tag{3.69}$$

where H^+_{ij} and H^-_{ij} form a time-independent orthonormal basis of helicity tensors with $m = \pm 2$ respectively and h_+ and h_- are their respective time-dependent coefficients [14].

Upon substituting equation (3.69) in equation (3.68), we find:

$$\ddot{h}_+ = -\frac{|k|^2}{a^2}h_+ - 3H\dot{h}_+. \tag{3.70}$$

This is the equation for gravitational waves in the FLRW universe [15].

One should recall at this point that the helicity tensor perturbation only affects the spatial metric. The graviton modes are therefore present in the

[14] We previously saw in Section 2.4.5 that we could construct such a basis

[15] We find a similar equation for h_-.

perturbation to the spatial metric.

This concludes our review of standard cosmological perturbation theory.

3.8 Summary

In this chapter we studied perturbations in the homogeneous and isotropic universe. We started by discussing the homogeneous and isotropic universe and formulated the corresponding metric and stress-energy tensors. Following that we calculated the Einstein equations. These were the background solutions. Next both - the metric and stress-energy - tensors were perturbed. The perturbations were decomposed into scalar, vector and tensor components. Following this we studied how a geometrical object responds to rotations in Fourier space. Based on the eigenvalues of these rotations we defined helicity scalars, helicity vectors and helicity tensors. We decomposed our perturbations into components of different helicities. It was noted that the helicity tensors were traceless and transverse to the three wave vector. Based on our findings from the previous chapter we expected that the helicity tensor perturbations to the metric will lead to the gravitational wave equation.

Following this decomposition in terms of helicities, we derived the Einstein equations to linear order in the perturbations. The helicity scalars exhibited non-trivial dynamics and were not analyzed further. The helicity vector modes vanished. Finally we found the propagation behavior for gravitational waves through the helicity tensor perturbation equations.

Chapter 4

Constrained Hamiltonian Dynamics and Gravity with Matter-Time

In this chapter we first introduce the Hamiltonian formalism which is a method of finding the dynamics of a system in terms of positions and momenta. The positions and momenta are used to calculate the Hamiltonian which is the energy of the system. We next note that some physical systems exhibit restrictions between their positions and momenta which we call constraints. In Section 4.2 we generalize the Hamiltonain formulation to include such physical systems.

In Section 4.3 we discuss the mathematical formulation of a field and generalize the Hamiltonian method to allow for us to study them. In Section

4.4 the Hamiltonian formulation of a scalar field in a curved spacetime is derived. We note that for a Minkowski spacetime, the dynamics of the scalar field are the same as the propagation equation for gravitational waves in a flat spacetime. A similar result is obtained for the scalar field in the FLRW universe.

In Section 4.5 we discuss the Hamiltonian formulation of GR. It is noted that the Hamiltonian in GR vanishes. We then - in Section 4.6 - consider a system of gravity and dust in its Hamiltonian form and equate the parameter time with dust. We find that under this specification (and within certain allowed rules) the Hamiltonian no longer vanishes. Following that we derive the dynamics of this system.

4.1 The Hamiltonian Formulation

We start with the action:

$$S = \int L(q_n, \dot{q}_n)dt \tag{4.1}$$

where S is the action, L is the Lagrangian and t is time. The Lagrangian is a function of the configuration variables q_n (which are also referred to as position) and their time derivatives $\dot{q}_n$ (also known as velocities). The positions and velocities specify the state of a system at any chosen time t_0 [8]. Note that the configuration variables are finite in number.

The momentum variables p_n are defined as:

$$p_n = \frac{\partial L}{\partial \dot{q}_n}. \tag{4.2}$$

The position and momentum variables are referred to as canonical variables and/or phase-space variables. For now we consider cases in which we can use equation (4.2) to express the velocities in terms of the phase-space variables. This implies that we are assuming the existence of an invertible map between the velocities and momenta. This assumption will be relaxed in Section 4.2.

The next step is to define the Hamiltonian as:

$$H = p_n \dot{q}_n - L \tag{4.3}$$

where the Lagrangian and the velocities have to be re-expressed in terms of the phase-space variables and there is a summation over n. We then re-express the action as:

$$S_c = \int [p_n \dot{q}_n - H] dt. \tag{4.4}$$

Expressed in this way the action is called the canonical action and the term $p_n \dot{q}_n$ is called the symplectic term.

Upon varying the action, extremizing it (i.e setting $\delta S_C = 0$) and comparing coefficients of the phase-space variables, we get the following equations of motion[1]:

$$\dot{q}_n = \frac{\partial H}{\partial p_n} \tag{4.5}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n}. \tag{4.6}$$

These are Hamilton's equations of motion. They represent the dynamics of a system in terms of the phase-space variables.

To rewrite Hamilton's equations differently we introduce the Poisson Bracket formalism. The Poisson Bracket (PB) for two functions $f(q_n, p_n)$ and $g(q_n, p_n)$ is defined as:

$$[f, g]_{PB} = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n}. \tag{4.7}$$

Hamilton's equations of motion may be re-expressed as:

$$\dot{q}_n = [q_n, H]_{PB} \tag{4.8}$$

$$\dot{p}_n = [p_n, H]_{PB}. \tag{4.9}$$

4.1.1 Example: Parameterized Particle in Flat Spacetime

The parameterized particle is defined to follow a path in spacetime that is parameterized by a single function λ. The action for a parameterized particle

[1] We assume that the variations of position at the end points of integration are zero.

is:

$$S = \int d\lambda \left(\sqrt{\eta_{ab} \dot{x}^a \dot{x}^b} \right) \tag{4.10}$$

where a dot denotes differentiation with respect to λ and $\dot{x}^a$ are the velocity terms.

We note that the form of the action stays unchanged if we change the parameter on the curve from λ to some function $f(\lambda)$. This change in parameter is called reparameterization. We note that the velocity terms change as:

$$\begin{aligned} \dot{x}^a &= \frac{dx^a}{d\lambda} \\ &= \frac{dx^a}{df}\frac{df}{d\lambda} \\ &= (x^a)' \dot{f} \end{aligned} \tag{4.11}$$

where the prime denotes differentiation with respect to the function $f(\lambda)$. We further note that:

$$df = \frac{df}{d\lambda} d\lambda. \tag{4.12}$$

This can be inverted to give:

$$d\lambda = \frac{df}{\dot{f}}. \tag{4.13}$$

By changing the parameter from λ to $f(\lambda)$ our action becomes:

$$S = \int df \left(\sqrt{\eta_{ab} (x^a)' (x^b)'} \right). \tag{4.14}$$

Therefore the action stays unchanged under this reparameterization. Because λ and $f(\lambda)$ correspond to time 't' from the definition of the Lagrangian in equation (4.1) we say that the theory is time reparameterization invariant (TRI).

We now proceed to find the Hamiltonian formulation[2]. The momentum is:

$$p_a = \frac{\eta_{ab}\dot{x}^b}{\sqrt{\dot{x}^c\dot{x}_c}} \tag{4.15}$$

and it is seen that inverting to get the velocity in terms of the momentum is not possible. We do however note that:

$$\eta^{ab}p_a p_b = 1. \tag{4.16}$$

Because inverting was problematic we find difficulty in re-expressing the Lagrangian in terms of the phase-space variables. However we note the following:

$$\begin{aligned} H &= p_a\dot{x}^a - L \\ &= \frac{\dot{x}_a\dot{x}^a}{\sqrt{\dot{x}_b\dot{x}^b}} - \sqrt{\dot{x}_c\dot{x}^c} \\ &= 0. \end{aligned} \tag{4.17}$$

The Hamiltonian for this system is zero.

In this subsection we first found that the action was invariant under repa-

[2]We have reverted back to λ.

rameterizations of λ. We encountered difficulty in expressing the velocities in terms of momenta. Then we found that the Hamiltonian was zero. To say more about the dynamics of the system we need to generalize the Hamiltonian formulation to include systems which have these symmetries; we do so in the next section.

4.2 Hamiltonian Formulation of Systems with Constraints

Previously in equation (4.2) we had assumed the existence of an invertible map between velocities and momenta. We now consider the case where the definition in equation (4.2) does not generate an invertible map. That happens - as we saw in the case for the parameterized particle - when there existed an independent relation between the momenta. We generalize beyond this and consider cases when there exist independent relations between the phase-space variables. We define these independent relations between the momentum and position variables to be the primary constraints of the Hamiltonian formalism [3][9][16][20]. The primary constraints are expressed as:

$$\phi_m(q, p) \approx 0 \tag{4.18}$$

where the subscript m represents the number of primary constraints. The $\approx$ sign is indicative of the constraints weakly equaling zero until they are enforced; that is when the $\approx$ sign gets replaced by a strong equality $=$. In the case of the parameterized particle the primary constraint is:

$$\phi = \eta^{ab} p_a p_b - 1 \approx 0. \tag{4.19}$$

When the velocities can be expressed in terms of the canonical variables, we write the Hamiltonian:

$$H = p_n \dot{q}_n - L. \tag{4.20}$$

This Hamiltonian is not unique. We can add to it any linear combination of the primary constraints and, because they are all zero, our theory will be unchanged. We use this freedom to write:

$$H^T = H + v_m \phi_m \tag{4.21}$$

where $v_m(t)$ are arbitrary functions of time and it will be assumed that they are not functions of the canonical variables. The reason for this specification will be discussed shortly. Hamilton's equations become:

$$\begin{aligned} \dot{q}_n &= \frac{\partial H}{\partial p_n} + v_m \frac{\partial \phi_m}{\partial p_n} \\ \dot{p}_n &= -\frac{\partial H}{\partial q_n} - v_m \frac{\partial \phi_m}{\partial q_n}. \end{aligned} \tag{4.22}$$

The time derivative of a function $g(q_n, p_n)$ is:

$$\dot{g} = \frac{\partial g}{\partial q_n}\dot{q_n} + \frac{\partial g}{\partial p_n}\dot{p_n}. \tag{4.23}$$

We substitute equation (4.22) in the equation above and rewrite in terms of Poisson Brackets to get:

$$\dot{g} = [g, H + v_m \phi_m]_{PB}. \tag{4.24}$$

Since the constraints are functions of the phase-space variables it is a revealing exercise to substitute them in the equation above. The expression for $\dot{\phi}_m$ becomes:

$$\begin{aligned} \dot{\phi}_m &= [\phi_m, H + v_{m'}\phi_{m'}]_{PB} \\ &= [\phi_m, H]_{PB} + v_{m'}[\phi_m, \phi_{m'}]_{PB}. \end{aligned} \tag{4.25}$$

We consider the different scenarios that unfold.

1. The equation above may lead to:

$$\dot{\phi}_m = \alpha\phi_m + \beta\phi_n \tag{4.26}$$

   where α and β are constants. The right hand side is a linear combination of primary constraints each of which were defined as $\phi \approx 0$.

Therefore our constraint ϕ_m is preserved in time:

$$\dot{\phi}_m \approx 0. \tag{4.27}$$

This is called the consistency condition of first kind.

2. Equation (4.25) may also lead to a function on the phase-space variables:

$$\dot{\phi}_m = \chi(q, p). \tag{4.28}$$

If we want for the evolution of the system to be constrained then the constraints of the system should be preserved in time. In this case we specify χ to be another constraint:

$$\chi(q, p) \approx 0. \tag{4.29}$$

This is called the consistency condition of second kind. Under this specification $\dot{\phi}_m \approx 0$. Constraints generated in this manner are called secondary constraints. The key difference between primary and secondary constraints is that primary constraints arise from the definition of momentum while secondary constraints come from Hamilton's equations of motion.

We then check for the time evolution of χ. If it follows the consistency condition of first kind then there are no more constraints in the theory. If it follows the consistency condition of second kind then we have the option of specifying another constraint i.e $\xi \approx 0$. We can keep repeating this process of generating constraints this way. We can also stop investigating the system further if the number of constraints increase such that the degrees of freedom in the theory reduce significantly. By extension we cannot generate constraints in this manner indefinitely because eventually we will be left with no degrees of freedom in the theory.

3. We can also have equation (4.25) reducing to:

$$\dot{\phi} = \lambda \tag{4.30}$$

   where λ is a constant. We specify the Lagrangian such that we do not encounter this condition.

4. We may also get the condition:

$$\dot{\phi} = v_m(t). \tag{4.31}$$

   In this case (if we want our constraints to be preserved) we set the time dependent coefficient $v_m(t)$ to zero.

We next define any function $R(q_n, p_n)$ to be first-class if it has zero Poisson Bracket with all the constraints - both primary and secondary - which are denoted by ϕ_j.

$$[R, \phi_j]_{PB} \approx 0. \tag{4.32}$$

If this condition does not hold then R is second-class. One should additionally note that if R is first-class then its Poisson Bracket with ϕ_j strongly equals a linear combination of the constraints i.e:

$$[R, \phi_j]_{PB} = r_{jj'}\phi_{j'}. \tag{4.33}$$

It can also be checked that the Poisson Bracket of two first-class quantities is first-class. We only keep first-class constraints in our theory and remove all second class constraints[3]. This is because - as it will be seen next - first-class constraints are of physical significance.

We consider an example involving time evolution of a physical system. Starting with the initial state g_0 we check what the state is at a later time δt.

$$\begin{aligned} g(\delta t) &= g_0 + \dot{g}\delta t \\ &= g_0 + \delta t[g, H + v_m\phi_m]_{PB} \end{aligned} \tag{4.34}$$

where we assume that ϕ_m is a primary, first-class constraint[4]. We will extend the treatment to secondary first-class constraints shortly. Suppose we had chosen a different arbitrary variable w_m.

$$\begin{aligned} g(\delta t) &= g_0 + \dot{g}\delta t \\ &= g_0 + \delta t[g, H + w_m\phi_m]_{PB}. \end{aligned} \tag{4.35}$$

[3]For details on this point please consult Appendix A.1
[4]We derive $g(\delta t)$ to linear order in δt.

The difference between the two final states is given by:

$$\begin{aligned} \Delta g(\delta t) &= \delta t[g, (v_m - w_m)\phi_m]_{PB} \\ &= e_m[g, \phi_m]_{PB} \end{aligned} \tag{4.36}$$

where

$$e_m = \delta t(v_m - w_m). \tag{4.37}$$

The final physical state is the same and therefore the two different set of phase-space variables correspond to the same physical state. Changes in phase-space variables which correspond to the same state are called gauge transformations. They are brought about by taking a Poisson Bracket of g with $e_a\phi_a$. The primary first-class constraints therefore lead to changes in the phase-space variables that correspond to the same state.

As an additional step suppose we apply two of these transformations in succession: first with $e_a\phi_a$ and then with $f_{a'}\phi_{a'}$. Redoing this procedure with the order of the transformations reversed should not change the state.

1. Applying $e_a\phi_a$ and then $f_{a'}\phi_{a'}$ to g_0 gives:

$$g^{'} = g_0 + e_a[g, \phi_a]_{PB} + f_{a'}[g + e_a[g, \phi_a]_{PB}, \phi_{a'}]_{PB}. \tag{4.38}$$

2. Going the other way gives:

$$g^{''} = g_0 + f_{a'}[g, \phi_{a'}]_{PB} + e_a[g + f_{a'}[g, \phi_{a'}]_{PB}, \phi_a]_{PB}. \tag{4.39}$$

The difference upon treatment with the Jacobi Identity becomes[5]:

$$\Delta g = e_a f_{a'} [g, [\phi_a, \phi_{a'}]_{PB}]_{PB}. \tag{4.40}$$

Both states are the same physically thus $[\phi_a, \phi_{a'}]_{PB}$ also generates infinitesimal transformations that do not change the state. Note that this is also first-class. The only way this treatment is more general that the first one is that before we only had first-class primary constraints as the generators of gauge transformations. This treatment admits first-class secondary constraints also. Note that the Poisson Bracket of two first-class constraints is also first-class; that in turn may be primary or secondary constraints (or a combination of the both). Also note that the Poisson Bracket of two primary constraints can give a secondary constraints as shown by the consistency condition of second kind.

Therefore the constraints that are important from a physical standpoint are first-class constraints because they are the generators of gauge-transformations that leave the physical state unchanged [9] [17]. For physically observable quantities, $\Delta g = 0$; consequently the Poisson Bracket of g with first-class constraints is zero. This implies that the physical observables are first-class quantities [16]. If a function has a non-zero Poisson Bracket with the first-class constraints it is (second-class by definition and) not physical. With a first-class constraint one should pick a condition on the phase-space vari-

[5]The Jacobi Identity is: $[f, [g, h]_{PB}]_{PB} + [h, [f, g]_{PB}]_{PB} + [g, [h, f]_{PB}]_{PB} = 0$

ables and solve the constraint. This condition is called a gauge. Fixing a gauge is like fixing a coordinate system in phase-space. Fixing coordinates is not physical and therefore it should be ensured that the gauge picked is second-class with the constraint it is used to solve.

We summarize the results and discussions from this section:

1. From the definition of the momentum we cannot always invert velocities in terms of the phase-space variables. This non-invertibility is a consequence of independent relations of the phase-space variables which we called the primary constraints. We wrote them as $\phi(q,p) \approx 0$.

2. Enforcing that these conditions are preserved in time sometimes led to more constraints. These were called secondary constraints.

3. A function R was defined as first-class if it had a zero Poisson Bracket with all the constraints.

4. First-class constraints generated gauge transformations. These were transformations in the phase-space variables that left the physical state unchanged. Thus the important constraints from a physical standpoint are the first-class constraints.

5. Physical observables are first-class quantities.

4.2.1 Example: Revisiting the Parameterized Particle

We will now proceed with our treatment of the parameterized particle. We recall that equation (4.19) is the primary constraint in the theory:

$$\phi \equiv \eta^{ab} p_a p_b - 1 \approx 0 \tag{4.41}$$

and that the Hamiltonian is zero. The total Hamiltonian is:

$$H_T = u\phi \tag{4.42}$$

where $u(t)$ is an arbitrary function of time. We specify that our constraint should be preserved in time. We write the equation for $\dot{\phi}$.

$$\begin{aligned} \dot{\phi} &= [\phi, H]_{PB} \\ &= [\phi, u\phi]_{PB} \\ &= 0. \end{aligned} \tag{4.43}$$

Therefore the constraint is preserved in time by the consistency condition of first kind. Since there is only one constraint, it is first-class. Therefore the total Hamiltonian is just a first-class constraint. By extension the evolution this Hamiltonian generates via Poisson Brackets will be a gauge transformation. Quantities that are gauge-invariant - i.e that have a zero Poisson Bracket with H (and ϕ) - are the physical observables. We now find the evolution equations for the canonical variables.

$$\begin{aligned} \dot{p}_a &= [p_a, u\phi]_{PB} \\ &= u[p_a, \eta^{bc} p_b p_c - 1] \\ &= 0. \end{aligned} \tag{4.44}$$

Therefore the momentum does not evolve and is a constant. For the position variable we find:

$$\begin{aligned} \dot{x}^a &= [x^a, u\phi]_{PB} \\ &= u[x^a, \eta^{bc} p_b p_c - 1]_{PB} \\ &= 2up^a. \end{aligned} \tag{4.45}$$

The evolution of x^a is therefore a constant p^a times an arbitrary function of time. This is a gauge transformation. Therefore the position variable evolves via a gauge transformation.

We now summarize the salient features of the Hamiltonian formulation of the parameterized particle in flat spacetime:

1. The action was invariant under reparameterizations of λ.

2. The Hamiltonian was a first-class constraint.

3. Consequently evolution was a gauge transformation.

4.2.2 Example: A Newtonian Particle

Consider the canonical action:

$$S = \int dt \left[p\dot{q} - \frac{p^2}{2} - V(q) \right] \tag{4.46}$$

where a dot represents differentiation with respect to time t and V - which is a function of the configuration variable - is a potential. Hamilton's equations of motion are:

$$\dot{q} = p \tag{4.47}$$

$$\dot{p} = -\frac{dV}{dq} \tag{4.48}$$

where p is the momentum conjugate to q. We change the time as: $t \to \tau = f(t)$. Under this change the action becomes:

$$\begin{aligned} S &= \int \frac{df}{\dot{f}} \left[pq^{'} \dot{f} - \frac{p^2}{2} - V(q) \right] \\ &= \int df \left[pq^{'} - \frac{p^2}{2\dot{f}} - \frac{V(q)}{\dot{f}} \right] \end{aligned} \tag{4.49}$$

where a prime denotes differentiation with respect to $f(t)$. Therefore the action in equation (4.46) changes under arbitrary time transformations.

We now invent a new action[6] by extending the phase space as:

$$(q(t), p(t)) \to (q(s), p(s), \phi(s), p_\phi(s)) \,. \tag{4.50}$$

[6]This action is engineered specifically to show that it is invariant under arbitrary time transformations.

The new action is:

$$S^\star = \int ds \left[p\dot{q} + p_\phi \dot{\phi} - N(s)\left(p_\phi + \frac{p^2}{2} + V(q) \right) \right] \tag{4.51}$$

where a dot denotes differentiation with respect to s and N is an arbitrary function of s. Because we have extended the phase space we anticipate constraints between the degrees of freedom. We will verify this shortly.

The equations of motion are:

$$\dot{q} = Np \tag{4.52}$$

$$\dot{p} = -N\frac{dV}{dq} \tag{4.53}$$

$$\dot{\phi} = N \tag{4.54}$$

$$\dot{p}_\phi = 0 \tag{4.55}$$

$$H \equiv p_\phi + \frac{p^2}{2} + V(q) \approx 0. \tag{4.56}$$

Equation (4.54) indicates that the time derivative of ϕ is the arbitrary function N. Equation (4.56) indicates that the Hamiltonian of this system is constrained to vanish. Also since there is only one constraint, H is first class. We substitute equations (4.54) and (4.56) in equation (4.51).

$$S^\star = \int ds[p\dot{q} + p_\phi\dot{\phi} - \dot{\phi}H]. \tag{4.57}$$

Upon a change in the parameter from s to $u = f(s)$ we find that the action does not change:

$$\begin{aligned} S^\star &= \int \frac{df}{\dot{f}}[pq^{'}\dot{f} + p_\phi \phi^{'}\dot{f} - \phi^{'}\dot{f}H] \\ &= \int df[pq^{'} + p_\phi \phi^{'} - \phi^{'}H] \end{aligned} \tag{4.58}$$

where a prime denotes differentiation with respect to $f(s)$. We have verified that the action $S^\star$ in equation (4.57) is TRI.

Now we fix a gauge and solve the Hamiltonian constraint[7]. The gauge chosen is:

$$\xi \equiv \phi - t \approx 0. \tag{4.59}$$

This gauge is second-class with H:

$$\begin{aligned} [\xi, H]_{PB} &= \left[\phi - t, p_\phi + \frac{p^2}{2} + V(q)\right]_{PB} \\ &= [\phi, p_\phi]_{PB} \\ &= 1. \end{aligned} \tag{4.60}$$

Solving the constraint gives:

$$p_\phi = -\frac{p^2}{2} - V(q). \tag{4.61}$$

[7]We have reverted back to the parameter s.

The action in (4.57) becomes:

$$S^\star = \int ds \left[p\dot{q} - \frac{p^2}{2} - V(q) \right]. \tag{4.62}$$

Hamilton's equations of motion are:

$$\dot{q} = p \tag{4.63}$$

$$\dot{p} = -\frac{dV}{dq}. \tag{4.64}$$

Therefore by picking the time-gauge $\xi \equiv \phi - t \approx 0$ we got the same dynamics as we did in equations (4.47) and (4.48).

We reiterate the important points of this subsection:

1. We started with the action S in equation (4.46) and found that it was not invariant under the transformation $t \to \tau = f(t)$.

2. We extended the phase space and our canonical variables depended on the parameter s. We specified the new action $S^\star$ and noted that it was invariant under reparameterizations of s.

3. Extending the phase space guarantees constraints in the theory. Consequently we found that the Hamiltonian was constrained to vanish. We picked a time-gauge to solve the constraint and get the physical Hamiltonian. Also by enforcing the time-gauge we shrunk the phase-space such that it only contained the canonical pair (q, p).

4.3 Hamiltonian Formulation of Fields

A field generates a number or a tensor at each point in spacetime. For example a scalar field $\phi(x^a)$ generates a number at each point in spacetime and it can be used to model:

1. Matter densities in the universe,
2. Pressure variations in the oceans and
3. Temperature of the atmosphere.
4. Also recall that in Section 2.4.5 we were able to view our metric perturbations in terms of functions that return a number for each point in spacetime. They are also examples of scalar fields.

A vector field $E^a(x^b)$ gives a vector at each point in spacetime and it can be used to model the electric force due to a charged particle. Another example of a tensor field is the metric $g_{ab}(x^c)$.

For the Hamiltonian formalism of a field we separate spacetime into space and time. This is done by choosing a parameter time t on spacetime such that $t = constant$ gives spacelike surfaces Σ_t. We take the configuration variable of our Hamiltonian theory to be the field at a surface of constant time t_0 i.e $\psi^a(x^i, t_0)$. The velocity is its time derivative $\dot{\psi}(x^i, t_0)$. Both ψ and $\dot{\psi}$ specify the state of the system at a given time t_0. Because both ψ and $\dot{\psi}$ are continuous functions of x^i for a constant time t_0 there is an infinite number of degrees of freedom in the theory [8]. We assume the field to be zero at the boundaries of the surface.

To have a notion of evolution we define a vector field t^a such that:

$$t^a \nabla_a t = 1. \tag{4.65}$$

This vector field defines the same point in space at different instants of time. Thus evolution is given by the Lie derivative with respect to t^a. The Lie derivative for a tensor p^a_b with respect to t^a is defined as:

$$\pounds_t p^a_b = t^c p^a_{b;c} + p^a_c t^c_{;b} - p^c_b t^a_{;c}. \tag{4.66}$$

The time derivative of ψ^a is therefore given by its Lie derivative with respect to t^a:

$$\dot{\psi^a} = \pounds_t \psi^a. \tag{4.67}$$

We call t^a the time-flow vector field [10] [15].

We write the Lagrangian of a field ψ^a as an integral on Σ_t:

$$L = \int_{\Sigma_t} d^3x \bar{L}(\psi^a(t, x^i), \dot{\psi^a}(t, x^i)) \tag{4.68}$$

where $\bar{L}$ is called the Lagrangian density. The conjugate momenta are defined

[8]Previously, we developed the Hamiltonian theory for the case when the configuration variable - q_n evaluated at constant time - were finite in number.

as:

$$\pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = \frac{\partial \bar{L}}{\partial \dot{\psi}^a} \tag{4.69}$$

and they are also infinite in number. The Hamiltonian is also expressed as an integral:

$$H = \int_{\Sigma_t} d^3x \bar{H}(\psi^a, \pi_a) \tag{4.70}$$

where $\bar{H}$ is the Hamiltonian density over Σ_t and it is computed as:

$$\bar{H} = \pi_a \dot{\psi}^a - \bar{L}. \tag{4.71}$$

The canonical action is:

$$S_C = \int dt \int_{\Sigma_t} d^3x [\pi_a \dot{\psi}^a - \bar{H}] \tag{4.72}$$

where $\int_{\Sigma_t} d^3x[\pi_a \dot{\psi}^a]$ is the symplectic term. The fundamental Poisson Bracket between a field $\phi_a(t,x)$ and its conjugate momentum $\pi_b(t,y)$ is:

$$\begin{aligned} [\phi^a(t,x), \pi_b(t,y)]_{PB} &= \int_{\Sigma_t} d^3z \left[\frac{\delta \phi^a(t,x)}{\delta \phi^c(t,z)} \frac{\delta \pi_b(t,y)}{\delta \pi_c(t,z)} - \frac{\delta \phi^a(t,x)}{\delta \pi^c(t,z)} \frac{\delta \pi_b(t,y)}{\delta \phi_c(t,z)} \right] \\ &= \int_{\Sigma_t} \left[\delta^a_c \delta^c_b \delta^3(x-z) \delta^3(y-z) - 0 \right] \\ &= \delta^a_b \delta^3(x-y). \end{aligned} \tag{4.73}$$

Hamilton's equations are:

$$\dot{\psi}^a = [\psi^a, H]_{PB} \tag{4.74}$$

$$\dot{\pi}_a = [\pi_a, H]_{PB}. \tag{4.75}$$

4.4 Hamiltonian Formulation of a Scalar Field on a Curved Background

The action for a scalar field ϕ is:

$$S_\phi = -\frac{1}{2}\int d^4x[\sqrt{-g}g^{ab}\partial_a\phi\partial_b\phi]. \tag{4.76}$$

Note the inverse metric in the action. This should be expressed in a convenient way that facilitates the Hamiltonian formulation. Arnowitt, Deser and Misner (ADM) derived such an expression and it follows from the separation of spacetime into a parameter time and space. We recall that in this setting there was a time-flow (t^a) which satisfied $t^a\nabla_a t = 1$. We express the time-flow in terms of its components normal to Σ_t and tangential to Σ_t as:

$$t^a = Nn^a + N^a \tag{4.77}$$

where N is called the lapse, n^a is the unit timelike normal to the surface and N^a is called the shift and it is tangential to the surface. In this setting the interval between a point (t, x^a) on Σ_t and a point $(t+dt, x^a+dx^a)$ on Σ_{t+dt} is [18]:

$$ds^2 = -N^2dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \tag{4.78}$$

where q_{ab} is the metric on the surface Σ_t. Note that $q_{ab}n^a = N_a n^a = 0$ because the spatial metric and the shift both reside on the spatial surface. The total number of functions in the metric is ten: there are three functions in the shift, one in the lapse and six in the three-metric (because it is a symmetric three-tensor).

We can use the metric g_{ab} from the equation (4.78) and calculate the inverse metric g^{ab} using the relation $g^{ab}g_{bc} = \delta^a_c$. Alternatively we can consider a route in which we start with the general definition of the four-metric:

$$g_{ab} = q_{ab} - n_a n_b. \tag{4.79}$$

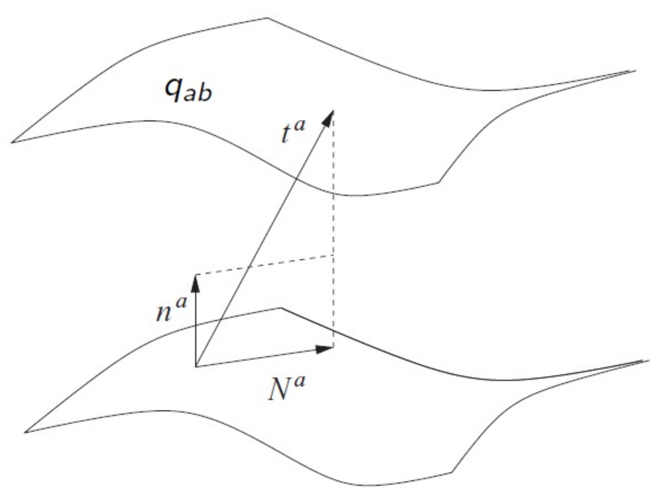

Figure 4.1: ADM Decomposition [1]

The inverse of the above is:

$$g^{ab} = q^{ab} - n^a n^b. \tag{4.80}$$

Using equation (4.77) we express the normal in terms of the time-flow and lapse:

$$n^a = \frac{t^a - N^a}{N}. \tag{4.81}$$

We next substitute n^a in equation (4.80) to get:

$$g^{ab} = -\frac{t^a t^b}{N^2} + \frac{2N^{(a}t^{b)}}{N^2} + q^{ab} - \frac{N^a N^b}{N^2}. \tag{4.82}$$

We can also check that:

$$\sqrt{-g} = N\sqrt{q}. \tag{4.83}$$

After substituting the inverse metric in the action in equation (4.76) we get the terms $t^a \partial_a \phi$. We note that the Lie derivative of the scalar field ϕ with respect to t^a is:

$$\pounds_t \phi = t^a \partial_a \phi. \tag{4.84}$$

We call all such terms $\dot{\phi}$ and they serve as the velocity terms in the Hamiltonian formalism. The Lagrangian density is:

$$\bar{L}^\phi = \frac{N\sqrt{q}}{2}\left[\frac{\dot{\phi}^2}{N^2} - \frac{2\dot{\phi}N^a\partial_a\phi}{N^2} - (q^{ab} - \frac{N^a N^b}{N^2})\partial_a\phi\partial_b\phi\right]. \tag{4.85}$$

The momentum conjugate to the field is given by the usual prescription:

$$\begin{aligned} p_\phi &= \frac{\partial \bar{L}}{\partial \dot{\phi}} \\ &= \frac{\sqrt{q}}{N}[\dot{\phi} - N^a \partial_a \phi]. \end{aligned} \tag{4.86}$$

Inverting for $\dot{\phi}$ gives:

$$\dot{\phi} = N^a \partial_a \phi + \frac{N p_\phi}{\sqrt{q}}. \tag{4.87}$$

The Lagrangian density in terms of the phase-space variables is:

$$\bar{L}^\phi = \frac{N\sqrt{q}}{2}\left[\frac{p_\phi^2}{q} - q^{ab}\partial_a\phi\partial_b\phi\right]. \tag{4.88}$$

We next find the Hamiltonian density to be:

$$\bar{H} = N\mathcal{H}^\phi - N^a C_a^\phi \tag{4.89}$$

where

$$\mathcal{H}^\phi = \frac{1}{2}\left(\frac{p_\phi^2}{\sqrt{q}} + \sqrt{q}q^{ab}\partial_a\phi\partial_b\phi\right) \tag{4.90}$$

and

$$C_a^\phi = p_\phi \partial_a \phi. \tag{4.91}$$

The canonical action is:

$$S_C^\phi = \int dt \int_{\Sigma_t} d^3x \left[p_\phi\dot{\phi} - N\mathcal{H}^\phi - N^a C_a^\phi\right]. \tag{4.92}$$

The phase space consists of the canonical pair of (ϕ, p_ϕ).

The equations of motion are:

$$\begin{aligned} \dot{\phi} &= N^a \partial_a \phi + \frac{N p_\phi}{\sqrt{q}} \\ \dot{p}_\phi &= \partial_b (p_\phi N^b + N \sqrt{q} q^{ab} \partial_a \phi). \end{aligned} \tag{4.93}$$

We now find how a scalar field propagates on a flat spacetime. We compare the ADM form of the metric in equation (4.78) to the Minkowski metric in equation (2.17). We set $N^a = 0$, $N = 1$ and $q^{ab} = e^{ab}$. The equations of motion are:

$$\begin{aligned} \dot{\phi} &= p_\phi \\ \dot{p}_\phi &= e^{ab} \partial_a \partial_b \phi. \end{aligned} \tag{4.94}$$

We now find the decoupled equation of motion for ϕ. We take another time derivative on the $\dot{\phi}$ equation and then re-express everything in terms of ϕ. We find that:

$$\ddot{\phi} = e^{ab} \partial_a \partial_b \phi. \tag{4.95}$$

Taking a spatial Fourier transform of the above gives:

$$\ddot{\tilde{\phi}} = -|\vec{k}|^2 \tilde{\phi}. \tag{4.96}$$

This equation above represents the dynamics of a scalar field in a flat spacetime. This is the same as the gravitational wave equation that we found in

Chapter 3 in equation (3.26).

We now check how a scalar field propagates in a homogeneous and isotropic universe. We compare the ADM form of the metric in equation (4.78) to the metric in equation (3.3) and set $N^a = 0$, $N = 1$ and $q^{ab} = \frac{e^{ab}}{a^2}$. We find that:

$$\begin{aligned} \dot{\phi} &= \frac{p_\phi}{a^3} \\ \dot{p}_\phi &= a e^{ab} \partial_a \partial_b \phi. \end{aligned} \tag{4.97}$$

The above in decoupled form is:

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi} + \frac{e^{ab}\partial_a\partial_b\phi}{a^2}. \tag{4.98}$$

We substitute the Hubble factor and take the spatial Fourier transform to find:

$$\ddot{\tilde{\phi}} = -3H\dot{\tilde{\phi}} - \frac{|\vec{k}|^2}{a^2}\tilde{\phi}. \tag{4.99}$$

This is how a scalar field propagates in a homogeneous and isotropic universe. This confirms that the metric functions that propagate as gravitational waves - equation (3.70) - can be expressed in terms of scalar fields.

4.5 Hamiltonian Formulation of General Relativity

The action for General Relativity is:

$$S^{GR} = \int d^4x\sqrt{-g}R \tag{4.100}$$

where R is the Ricci scalar for the full metric.

We substitute the ADM decomposition in $\sqrt{-g}$ and R[9]. The latter - after using the Gauss-Codacci equation - becomes:

$$R =^{(3)} R - K^2 + K^{ab}K_{ab} \tag{4.101}$$

where ${}^{(3)}R$ is the Ricci scalar on the spatial slice, K_{ab} is the extrinsic curvature of the spatial slice and K is its trace [10]. The extrinsic curvature K_{ab} describes how the surface Σ_t is embedded in the spacetime and it is defined as:

$$K_{ab} = \frac{\pounds_n q_{ab}}{2}. \tag{4.102}$$

We will now express the extrinsic curvature in a different algebraic form. Substituting the definition of the Lie derivative (4.66) in the equation above gives:

$$K_{ab} = \frac{1}{2}[n^c q_{ab;c} + 2n^c_{;(b}q_{a)c}]. \tag{4.103}$$

[9]We explain how one can do calculus on Σ_t in Appendix A.2. This Appendix chapter also has a detailed account of the ADM foliation.

We then multiply and divide the right hand side by N to get:

$$\begin{aligned} K_{ab} &= \frac{1}{2N}[(Nn^c)q_{ab;c} + 2Nn^c_{;(b}q_{a)c}] \\ &= \frac{1}{2N}[(Nn^c)q_{ab;c} + 2(Nn^c)_{;(b}q_{a)c} - 2N_{,(b}q_{a)c}n^c]. \end{aligned} \tag{4.104}$$

The third term in the last equation goes to zero because of orthonormality.

Recall that from the $3+1$ decomposition the normal may be expressed as:

$$Nn^c = t^c - N^c. \tag{4.105}$$

Upon substituting the ADM decomposition for the normal equation (4.104) becomes:

$$\begin{aligned} K_{ab} &= \frac{1}{2N}[t^c q_{ab;c} + 2t^c_{;(b}q_{a)c} - (N^c q_{ab;c} + 2N^c_{;(b}q_{a)c})] \\ &= \frac{1}{2N}[\pounds_t q_{ab} - \pounds_N q_{ab}] \end{aligned} \tag{4.106}$$

where the definition for the Lie derivative with respect to t^a and N^a has been used in going from the first equation to the next. The first term on the right hand side may be rewritten as $\dot{q}_{ab}$. For the second term, realize that the Lie derivative is being taken with respect to a spatial vector and the object being operated upon is also a spatial object. By extension one may use the spatial covariant derivative D_a in the definition of the Lie derivative[10]:

$$\begin{aligned} K_{ab} &= \frac{1}{2N}[\dot{q}_{ab} - N^c D_c q_{ab} - 2q_{c(a}D_{b)}N^c] \\ &= \frac{1}{2N}[\dot{q}_{ab} - 2D_{(b}N_{a)}]. \end{aligned} \tag{4.107}$$

Therefore the velocity terms in the action enter through the extrinsic curvature.

The Lagrangian density in GR is:

$$\begin{aligned} \bar{L}^{GR} &= N\sqrt{q}(^{(3)}R - K^2 + K^{ab}K_{ab}) \\ &= N\sqrt{q}(^{(3)}R - q^{ab}K_{ab}q^{cd}K_{cd} + q^{ac}q^{bd}K_{cd}K_{ab}) \end{aligned} \tag{4.108}$$

and it is understood that the velocity terms $\dot{q}_{ab}$ enter through K_{ab}. The three-momentum π^{ab} is calculated via the usual prescription:

$$\begin{aligned} \pi^{ab} &= \frac{\partial \bar{L}^{GR}}{\partial \dot{q}_{ab}} \\ &= \frac{\partial \bar{L}^{GR}}{\partial K_{ab}} \frac{\partial K_{ab}}{\partial \dot{q}_{ab}} \\ &= \sqrt{q}(K^{ab} - q^{ab}K). \end{aligned} \tag{4.109}$$

To re-express the velocity and the Lagrangian density in terms of the phase-space variables we start by computing the trace of the three-momentum:

$$\begin{aligned} \pi &= \pi^{ab}q_{ab} \\ &= -2\sqrt{q}K. \end{aligned} \tag{4.110}$$

[10]For details on the spatial covariant derivative consult Appendix A.2.

Then we solve for K using the above equation:

$$K = -\frac{\pi}{2\sqrt{q}}. \tag{4.111}$$

Next we solve for K_{ab} using (4.109) to get:

$$K_{ab} = \frac{1}{\sqrt{q}}[\pi_{ab} - \frac{\pi}{2}q_{ab}]. \tag{4.112}$$

We recall from (4.107) that:

$$K_{ab} = \frac{1}{2N}[\dot{q}_{ab} - 2D_{(a}N_{b)}]. \tag{4.113}$$

By equating the two equations above, we find the velocity in terms of the phase-space variables to be:

$$\dot{q}_{ab} = \frac{2N}{\sqrt{q}}(\pi_{ab} - \frac{\pi}{2}q_{ab}) + 2D_{(b}N_{a)}. \tag{4.114}$$

The Lagrangian density in terms of the phase-space variables is:

$$\bar{L}^{GR} = N\sqrt{q}^{(3)}R + \frac{N\pi^{ab}\pi_{ab}}{\sqrt{q}} - \frac{N\pi^2}{2\sqrt{q}}. \tag{4.115}$$

We next compute for the Hamiltonian density:

$$\bar{H}^{GR} = N\mathcal{H}^{GR} + N^a C_a^{GR} \tag{4.116}$$

where

$$\mathcal{H}^{GR} = -\sqrt{q}^{(3)}R + \frac{\pi_{ab}\pi^{ab}}{\sqrt{q}} - \frac{\pi^2}{2\sqrt{q}} \tag{4.117}$$

and

$$C_a^{GR} = -2D_b\pi_a^b. \tag{4.118}$$

The canonical action for General Relativity is:

$$S_C^{GR} = \int dt \int_{\Sigma_t} d^3x[\pi^{ab}\dot{q}_{ab} - N\mathcal{H}^{GR} - N^a C_a^{GR}]. \tag{4.119}$$

We can now vary the action with respect to the metric functions. Vary the action with respect to N to get $\mathcal{H}^{GR} \approx 0$; this is the Hamiltonian constraint. Do the same with N^a to get $C_a^{GR} \approx 0$; that is called the diffeomorphism constraint. Both, the Hamiltonian and diffeomorphism constraints, are first-class constraints [15]. Therefore the total Hamiltonian is constrained to zero. We will find the equations of motion for the phase-space variables (q_{ab}, π^{ab}) shortly.

It is instructive to count the degrees of freedom in phase-space. We have a combined 12 degrees of freedom in q_{ab} and π^{ab}. There are 4 first class constraints that come with 4 gauge choices. These reduce the degrees of freedom in the theory to $12 - 4 - 4 = 4$. Therefore the theory has 4 gravitational degrees of freedom in phase-space.

4.6 The Canonical Dust-Time Gauge

4.6.1 Action and Hamiltonian Theory

We consider the action of General Relativity, coupled to a massive, pressure-less and time-like dust [14]:

$$S = \int d^4x\sqrt{-g}R - \frac{1}{2}\int d^4x\sqrt{-g}m(g^{ab}\partial_a\phi\partial_b\phi + 1). \tag{4.120}$$

The second action is that of dust which is represented by the scalar field ϕ. The function $m(x^a)$ represents the mass density of dust; it also enforces the gradient of the field to be timelike. We check this by varying the action with respect to m and extremizing it:

$$\begin{aligned} \frac{\delta S}{\delta m} &= 0 \\ g^{ab}\partial_a\phi\partial_b\phi + 1 &= 0 \\ g^{ab}\partial_a\phi\partial_b\phi &= -1. \end{aligned} \tag{4.121}$$

In Section 3.1.1 we discussed that dust represents matter that does not interact with other matter. Therefore the action in equation (4.120) represents a theory of gravity with matter that has zero pressure.

We use the ADM foliation to derive the Hamiltonian formulation of this system. The Lagrangian density is:

$$\bar{L}^D = -\frac{N\sqrt{q}m}{2}\left[1 - \frac{\dot{\phi}^2}{N^2} + \frac{2\dot{\phi}N^a\partial_a\phi}{N^2} + q^{ab}\partial_a\phi\partial_b\phi - \frac{N^aN^b\partial_a\phi\partial_b\phi}{N^2}\right]. \tag{4.122}$$

The momentum p_ϕ is calculated via the prescription:

$$\begin{aligned} p_\phi &= \frac{\partial\bar{L}^D}{\partial\dot{\phi}} \\ &= \frac{m\sqrt{q}}{N}[\dot{\phi} - N^a\partial_a\phi]. \end{aligned} \tag{4.123}$$

We invert to get $\dot{\phi}$ in terms of the phase-space variables.

$$\dot{\phi} = \frac{Np_\phi}{\sqrt{q}m} + N^a\partial_a\phi. \tag{4.124}$$

The Lagrangian density in terms of the canonical variables is:

$$\bar{L}^D = -\frac{N\sqrt{q}m}{2}\left[1 - \frac{p_\phi^2}{qm^2} + q^{ab}\partial_a\phi\partial_b\phi\right]. \tag{4.125}$$

The Hamiltonian density is:

$$\bar{H}^D = \frac{N}{2}\left[\frac{p_\phi^2}{m\sqrt{q}} + m\sqrt{q}(q^{ab}\partial_a\phi\partial_b\phi + 1)\right] + N^ap_\phi\partial_a\phi. \tag{4.126}$$

The canonical action for the gravity-dust system is:

$$S_C = \int dt \int_{\Sigma_t} d^3x[\pi^{ab}\dot{q}_{ab} + p_\phi \dot{\phi} - N(\mathcal{H}^{GR} + \mathcal{H}^D) - N^a(C_a^{GR} + C_a^D)] \tag{4.127}$$

where

$$\mathcal{H}^D = \frac{1}{2}[\frac{p_\phi^2}{m\sqrt{q}} + m\sqrt{q}(q^{ab}\partial_a\phi\partial_b\phi + 1)] \tag{4.128}$$

and

$$C_a^D = p_\phi \partial_a \phi. \tag{4.129}$$

$\mathcal{H}^{GR}$ and C_a^{GR} have been calculated in the previous section. Our Hamiltonian constraint is:

$$\mathcal{H}^{GR} + \mathcal{H}^D \approx 0 \tag{4.130}$$

and our diffeomorphism constraint is:

$$C_a^{GR} + C_a^D \approx 0. \tag{4.131}$$

We vary the action with respect to m and extremize it to find:

$$m = \pm \frac{p_\phi}{\sqrt{q(q^{ab}\partial_a\phi\partial_b\phi + 1)}}. \tag{4.132}$$

We use this expression to express our theory without m. As a result $\mathcal{H}^D$ becomes:

$$\mathcal{H}^D = \pm p_\phi \sqrt{q^{ab}\partial_a\phi\partial_b\phi + 1}. \tag{4.133}$$

Finally we count the degrees of freedom (in phase-space) in the theory. There are:

1. 6 degrees of freedom in q_{ab} and π^{ab} each.

2. 1 degree of freedom in ϕ and p_ϕ each. The total number of degrees of freedom is therefore 14.

3. 3 first-class constraints in the diffeomorphism constraint and one in Hamiltonian constraint. The total number of first-class constraints is 4.

4. To solve the 4 first-class constraints we pick 4 gauges.

5. The remaining degrees of freedom are $14 - 8 = 6$. Recall from the previous section that 4 of these remaining degrees of freedom are gravitational. Therefore the other 2 are scalar.

4.6.2 Dust-Time Gauge

In this subsection we solve the Hamiltonian constraint by picking the dust-time gauge. The dust-time gauge is defined to "equate physical time with level values of the scalar field" i.e [12]:

$$\lambda \equiv \phi - t \approx 0. \tag{4.134}$$

It is checked that the gauge is second-class with the Hamiltonian constraint:

$$\begin{aligned}[\lambda, \mathcal{H}]_{PB} &= [\phi - t, \mathcal{H}^D + \mathcal{H}^{GR}]_{PB} \\ &= [\phi, \mathcal{H}^D]_{PB} \\ &= \pm[\phi, p_\phi \sqrt{q^{ab}\partial_a\phi\partial_b\phi + 1}]_{PB} \\ &\neq 0\end{aligned} \tag{4.135}$$

where we have substituted (4.133) in going from the second to the third line. Since the above is not zero the dust-time gauge is second-class and hence can be used to solve the Hamiltonian constraint.

As a consequence of picking the dust-time gauge we note the following simplifications:

1. Equation (4.132) for m becomes:

$$m = \pm\frac{p_\phi}{\sqrt{q}}. \tag{4.136}$$

   This is because in the dust-time gauge:

$$\begin{aligned}q^{ab}\partial_a\phi\partial_b\phi &= q^{ab}\partial_a t\partial_b t \\ &= 0.\end{aligned} \tag{4.137}$$

2. By extension equation (4.133) for $\mathcal{H}^D$ also becomes simpler:

$$\mathcal{H}^D = \pm p_\phi. \tag{4.138}$$

3. Lastly the diffeomorphism constraint for dust vanishes:

$$\begin{aligned} C_a^D &= p_\phi \partial_a \phi \\ &= 0. \end{aligned} \tag{4.139}$$

We specify that the gauge should be preserved under evolution:

$$\begin{aligned} \phi &\approx t \\ \dot{\phi} &\approx 1. \end{aligned} \tag{4.140}$$

We write Hamilton's equation for $\dot{\phi}$ and enforce the condition above:

$$\dot{\phi} = \left[\phi, \int_{\Sigma_t} d^3x (N\mathcal{H} + N^a C_a)\right]_{PB} \approx 1. \tag{4.141}$$

This simplifies to:

$$\begin{aligned} \left[\phi, \int_{\Sigma_t} d^3x (N\mathcal{H}^D)\right]_{PB} &\approx 1 \\ \left[\phi, \int_{\Sigma_t} d^3x (N p_\phi)\right]_{PB} &\approx \pm 1 \end{aligned} \tag{4.142}$$

where we have used equation (4.138) in going from the first line to the second. The lapse is[11]:

$$N = \pm 1. \tag{4.143}$$

We choose $N = 1$ to indicate forward evolution in time. This fixes $\mathcal{H}^D = p_\phi$.

[11]Since we have solved the Poisson Brackets we can replace the $\approx$ by a strong equality $=$.

Consequently the ADM metric in the dust time gauge is:

$$ds^2 = -dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \tag{4.144}$$

We now solve the Hamiltonian constraint to get:

$$\begin{aligned} \mathcal{H}^D + \mathcal{H}^{GR} &= 0 \\ p_\phi &= -\mathcal{H}^{GR}. \end{aligned} \tag{4.145}$$

The canonical action thus becomes:

$$S_C = \int dt \int_{\Sigma_t} d^3x[\pi^{ab}\dot{q}_{ab} - \mathcal{H}^{GR} - N^a C_a^{GR}] \tag{4.146}$$

with the total Hamiltonian not constrained to be zero anymore.

Hamilton's equations are:

$$\begin{aligned} \dot{q}_{ab} &= \left[q_{ab}, \int d^3x[\mathcal{H}^{GR} + N^a C_a^{GR}]\right]_{PB} \\ \dot{\pi}^{ab} &= \left[\pi^{ab}, \int d^3x[\mathcal{H}^{GR} + N^a C_a^{GR}]\right]_{PB}. \end{aligned} \tag{4.147}$$

To find the evolution equations for the phase-space variables we will identify where the variation will act and list the corresponding results.

1. The first terms that shall be checked is the inverse metric q^{ab}:

$$\delta(q^{ab}) = -q^{ac}q^{bd}\delta(q_{bc}). \tag{4.148}$$

2. The square root of the determinant becomes:

$$\delta(\sqrt{q}) = \frac{\sqrt{q}q^{ab}\delta(q_{ab})}{2}. \tag{4.149}$$

3. For the Christoffel symbol, the variation looks like:

$$\delta(\Gamma^a_{bc}) = \frac{q^{ad}}{2}[D_b(\delta q_{cd}) + D_c(\delta q_{bd}) - D_d(\delta q_{cb})]. \tag{4.150}$$

4. The Ricci tensor when varied gives:

$$\delta(R_{ab}) = -D_b(\delta\Gamma^d_{da}) + D_d(\delta\Gamma^d_{ab}). \tag{4.151}$$

   This is the Palatini Identity [1].

5. And now we can check for the variation of the Ricci scalar:

$$\delta R = R_{ab}(\delta q^{ab}) + q^{ab}(\delta R_{ab}). \tag{4.152}$$

6. The contraction of the momentum with itself, when varied gives:

$$\delta(\pi^{ab}\pi_{ab}) = 2[\pi_{ab}(\delta\pi^{ab}) + \pi^c_b\pi^{ab}(\delta q_{ac})]. \tag{4.153}$$

7. Varying the trace of the momentum gives:

$$\delta(\pi) = q_{ab}(\delta\pi^{ab}) + \pi^{ab}(\delta q_{ab}). \tag{4.154}$$

8. Also for book-keeping, it becomes convenient to check for the following:

$$\delta\pi_a^b = \pi^{bc}(\delta q_{ac}) + q_{ac}(\delta\pi^{bc}). \tag{4.155}$$

From here, it is a matter of varying the action, substituting for the variation of different terms and extremizing the action to get the Einstein field equations in the canonical dust-time gauge. The evolution equations in the canonical dust-time gauge are:

$$\dot{q}_{ab} = \frac{2}{\sqrt{q}}\pi_{ab} - \frac{\pi q_{ab}}{\sqrt{q}} + 2D_{(b}N_{a)} \tag{4.156}$$

$$\begin{aligned}\dot{\pi}^{ab} = \sqrt{q}(\frac{^{(3)}Rq^{ab}}{2} -^{(3)} R^{ab}) + \frac{q^{ab}}{2\sqrt{q}}(\pi^{cd}\pi_{cd} - \frac{\pi^2}{2}) \\ - \frac{2}{\sqrt{q}}(\pi_d^a\pi^{bd} - \frac{\pi\pi^{ab}}{2}) - 2\pi^{d(b}D_dN^{a)} + D_d(\pi^{ab}N^d).\end{aligned} \tag{4.157}$$

The diffeormorphism constraint is:

$$D_b\pi^{ab} = 0. \tag{4.158}$$

4.7 Summary

The Hamiltonian formalism was introduced for finite degrees of freedom. In doing the Hamiltonian formulation of the parameterized particle it was found that the Hamiltonian was zero. This motivated the generalization of the Hamiltonian formulation to include constraints. We identified first-class constraints as particularly important as they were generators of gauge transformations. We studied another time reparameterization invariant theory and learned that picking a time gauge and solving the Hamiltonian constraint leads to a physical Hamiltonian.

We next reviewed fields and noted that they have infinite degrees of freedom. Since the Hamiltonian formulation was introduced for finite degrees of freedom, we generalized it to include fields. Following that we derived the Hamiltonian formulation of a scalar field on a curved background. We used the ADM decomposition of spacetime to express our ten metric functions. We found that for Minkowski spacetime the dynamics of the scalar field were identical to the propagation equations of gravitational waves. We found a similar result for the FLRW universe. Following this we studied the Hamiltonian formulation of GR and noted that the Hamiltonian is constrained to vanish.

We then considered the action of gravity with a pressure-less and time-like dust and derived the Hamiltonian formulation of this system. We solved the Hamiltonian constraint of this system with the gauge that dust is time. In enforcing this gauge the lapse got fixed to 1 and the Hamiltonian was no longer constrained to vanish. We concluded this section by writing the equations of motion of the three-metric and the three-momentum along with the diffeomorphism constraint in the dust-time gauge.

Chapter 5

Linearized Hamiltonian Theory in Cosmology with Dust-Time

In this chapter we start by specifying the three-metric, its conjugate momentum and the shift in terms of backgrounds and perturbations. In Section 5.1 we derive the evolution equations and the diffeomorphism constraint to linear order in the perturbations. In Section 5.2 the perturbations are expressed in terms of spatial Fourier modes. In Section 5.3 the results are decomposed into scalar, vector and tensor components. In Section 5.4 we fix gauges, solve the first-class constraints and analyze our results.

This method has been tested in a Minkowski spacetime with the result that the graviton modes satisfy the wave equation on a flat background [12]. The new research conducted for this thesis is an application of this method to FLRW cosmology. This chapter contains our calculations, findings and analysis.

5.1 Perturbed Equations of Motion

5.1.1 Hamiltonian Dynamics of FLRW Cosmology

Recall from Section 4.5 that the phase-space of GR contains the canonical pair q_{ab} and π^{ab}. Also recall that in Section 4.4 we had discussed the metric of FLRW cosmology in terms of the ADM variables. We had set $N^a = 0$ and $q_{ab}(t) = a^2(t)e_{ab}$ [1]. The time-dependence in q_{ab} comes from the scale factor. Therefore the scale factor is the configuration variable in FLRW cosmology. The symplectic term in the canonical action should be:

$$\int d^3 x p \dot{a} \tag{5.1}$$

where p is the momentum conjugate to a. The momentum p should come from π^{ab}. We consider the simplest form π^{ab} can take:

$$\pi^{ab}(t) = X(t)e^{ab} \tag{5.2}$$

[1] In the dust-time gauge the lapse is fixed to 1

where X is an unknown variable. We will use the fact that $\pi^{ab}\dot{q}_{ab}$ from equation (4.146) reduces to $p\dot{a}$ to calculate the form of X.

$$\begin{aligned}\pi^{ab}\dot{q}_{ab} &= Xe^{ab}2a\dot{a}e_{ab} \\ &= (6aX)\dot{a}.\end{aligned} \tag{5.3}$$

The term in brackets should be p. Therefore the momentum conjugate to q_{ab} is:

$$\pi^{ab}(t) = \frac{p(t)}{6a(t)}e^{ab}. \tag{5.4}$$

We now calculate the Hamiltonian dynamics of FLRW cosmology. Using equation (4.117) we find that the Hamiltonian is:

$$\mathcal{H}^{GR} = -\frac{p^2}{24a}. \tag{5.5}$$

It leads to the following equations of motion for a and p.

$$\dot{a} = -\frac{p}{12a} \tag{5.6}$$

$$\dot{p} = -\frac{p^2}{24a^2}. \tag{5.7}$$

The decoupled equation for a is:

$$\ddot{a} = -\frac{\dot{a}^2}{2a}. \tag{5.8}$$

The solution for $a(t)$ is:

$$\frac{2}{3}a^{\frac{3}{2}} - C_1 t - C_2 = 0 \tag{5.9}$$

where C_1 and C_2 are constants of integration[2]. If we set $C_1 = 0$ then (5.9) leads to a constant scale factor:

$$a(t) = C^{\frac{2}{3}} \tag{5.10}$$

where C is another constant. If the scale factor is a constant a_0, then the three-metric is:

$$q_{ab} = a_0^2 dx^2 + a_0^2 dy^2 + a_0^2 dz^2. \tag{5.11}$$

If we make a coordinate transformation of the type $d\tilde{x}^a = a_0 dx^a$, then the three-metric becomes:

$$q_{ab} = e_{ab} d\tilde{x}^a d\tilde{x}^b. \tag{5.12}$$

This is the Euclidean three-metric. Also if the scale factor is a constant then $\dot{a} = p = 0$. Therefore the solution in (5.10) implies that the universe is static and does not expand. This is not a solution of interest. However, if we set $C_2 = 0$ then (5.9) leads to:

$$a(t) = Dt^{\frac{2}{3}} \tag{5.13}$$

where D is a constant.

[2] I used Maple to calculate this.

5.1.2 First Order Perturbation Equations of Motion

We specify the full perturbed quantities as:

$$q_{ab}(t,\vec{x}) = a(t)^2 e_{ab} + h_{ab}(t,\vec{x}) \tag{5.14}$$

$$\pi^{ab}(t,\vec{x}) = \frac{p(t)}{6a(t)} e^{ab} + p^{ab}(t,\vec{x}) \tag{5.15}$$

$$N^a(t,\vec{x}) = \xi^a(t,\vec{x}). \tag{5.16}$$

In these equations h_{ab} is the perturbation to the three-metric, p^{ab} is that to the three-momentum and ξ^a is the perturbation to the shift. We note that the perturbation to the three metric in equation (5.14) is not multiplied by the scale factor whereas in standard cosmological perturbation theory the tensor perturbation was multiplied by a^2; see equation (3.17). This is by design and will be justified shortly. Note that if we want to match results from the two approaches the rescaling should be accounted for at some stage.

We substitute the full q_{ab}, π^{ab} and N^a in the equations of motion and diffeomorphism constraint. The result is truncated at linear order in the perturbations. The evolution equations for the perturbations to linear order are:

$$\dot{h}_{ab} = 2ae_{ac}e_{bd}p^{cd} - ap^{cd}e_{cd}e_{ab} + \frac{p}{6a^2}h_{ab} - \frac{p}{6a^2}e^{cd}h_{cd}e_{ab} + 2a^2 e_{c(a}\xi^c_{,b)} \tag{5.17}$$

$$
\begin{aligned}
\dot{p}^{ab} = &-\frac{p}{6a^2}p^{ab} + \frac{p}{12a^2}e_{cd}p^{cd}e^{ab} - \frac{5p^2}{144a^5}e^{ac}e^{bd}h_{cd} \\
&+ \frac{p^2}{72a^5}e^{cd}h_{cd}e^{ab} - \frac{p}{3a}e^{c(a}\xi^{b)}_{,c} + \frac{p}{6a}e^{ab}\xi^{c}_{,c} \\
&+ \frac{e^{ij}}{2a^3}(\frac{e^{ab}e^{cd}}{2} - e^{ac}e^{bd})(h_{dj,ci} + h_{ci,dj} - h_{cd,ij} - h_{ij,cd})
\end{aligned}
\tag{5.18}
$$

and the linearized diffeomorphism constraint is:

$$
p^{ab}_{,b} + \frac{p}{12a^3}(e^{ad}e^{bc} + e^{ac}e^{bd})(h_{db,c} + h_{cd,b} - h_{cb,d}) = 0. \tag{5.19}
$$

For details on these calculations please consult Appendix A.3.

5.2 Spatially Fourier Transformed Results

Recall from Section 3.3.1 that the spatial Fourier transform allows us to view a function of space as plane waves along with their coefficients. That permits the required decomposition into scalar, vector and tensor (SVT) components[3,4]. These SVT components are what we intend to analyze eventually.

We expand the perturbations in modes of flat space Laplacian (plane

[3] In this chapter we only discuss decomposition of three-tensors in a helicity basis.

[4] Hereon we do not use the helicity prefix in this Chapter. When discussing - for example - the scalar perturbations, it should be understood that the reference is to helicity scalars.

waves) as:

$$h_{ab}(t,\vec{x}) = \int d^3k[e^{i\vec{k}.\vec{x}}\tilde{h}_{ab}(t,\vec{k})]$$
$$p^{ab}(t,\vec{x}) = \int d^3k[e^{i\vec{k}.\vec{x}}\tilde{p}^{ab}(t,\vec{k})] \tag{5.20}$$
$$\xi^{a}(t,\vec{x}) = \int d^3k[e^{i\vec{k}.\vec{x}}\tilde{\xi}^{a}(t,\vec{k})].$$

We substitute the above wherever there is a perturbation term. If there is a spatial derivative acting on the perturbation then it will pull down a factor of ik_a and a time derivative will act on the coefficient of the plane wave. Peeling off the common factors then gives the spatial Fourier transform of the equations of motion for the perturbations. These equations are:

$$\dot{\tilde{h}}_{ab} = 2ae_{ac}e_{bd}\tilde{p}^{cd} - a\tilde{p}^{cd}e_{cd}e_{ab} + \frac{p}{6a^2}\tilde{h}_{ab} - \frac{p}{6a^2}e^{cd}\tilde{h}_{cd}e_{ab} + 2ia^2e_{c(a}\tilde{\xi}^{c}k_{b)} \tag{5.21}$$

$$\begin{aligned}\dot{\tilde{p}}^{ab} &= -\frac{p}{6a^2}\tilde{p}^{ab} + \frac{p}{12a^2}e_{cd}\tilde{p}^{cd}e^{ab} - \frac{5p^2}{144a^5}e^{ac}e^{bd}\tilde{h}_{cd} \\ &+ \frac{p^2}{72a^5}e^{cd}\tilde{h}_{cd}e^{ab} - i\frac{p}{3a}e^{c(a}\tilde{\xi}^{b)}k_c + i\frac{p}{6a}e^{ab}\tilde{\xi}^{c}k_c \\ &- \frac{e^{ij}}{2a^3}(\frac{e^{ab}e^{cd}}{2} - e^{ac}e^{bd})(\tilde{h}_{dj}k_ck_i + \tilde{h}_{ci}k_dk_j - \tilde{h}_{cd}k_ik_j - \tilde{h}_{ij}k_ck_d).\end{aligned} \tag{5.22}$$

The spatial Fourier transform of the linearized diffeomorphism constraint is:

$$\tilde{p}^{ab}k_b + \frac{p}{12a^3}(e^{ad}e^{bc} + e^{ac}e^{bd})(\tilde{h}_{db}k_c + \tilde{h}_{cd}k_b - \tilde{h}_{cb}k_d) = 0. \tag{5.23}$$

5.3 Decomposing the Perturbations

5.3.1 The Basis Elements

In the three-dimensional Fourier Space we have symmetric three-dimensional matrices $\tilde{h}_{ab}$ and $\tilde{p}^{ab}$. A symmetric three-dimensional matrix has six independent entries; we therefore need an orthonormal set of six symmetric matrices to serve as a basis. There are many such basis sets we can construct. What is needed is a basis that gives a certain meaning to the components. For example we found in Chapter 2 that the graviton modes propagate via disturbances perpendicular to the plane of propagation and that they are also traceless. Therefore to extract these modes, one must decompose the equations using basis elements that are traceless and transverse.

The construction of this basis was actually started in Section 3.3.2. We want for our basis to be time-independent. Since $e^a_\pm$ and e^a_3 are time-independent our basis is automatically time-independent. We also want our basis to be orthonormal. However we first need a definition for orthonormality. An operation is required that multiplies two matrices and takes out a number from the product. Possible candidates are the trace and the determinant; this project will use the trace to define orthonormality as such[5].

$$M^I_{ab} M^{ab}_J = \delta^I_J. \tag{5.24}$$

[5]Recall we used a similar procedure to construct a basis in Chapter 2. We used that basis to decompose the traceless transverse metric perturbation.

The following M matrices are used for the SVT decomposition:

$$M_1^{ab} = \frac{1}{\sqrt{3}}(2e_+^{(a}e_-^{b)} + e_3^a e_3^b) \tag{5.25}$$

$$M_2^{ab} = \sqrt{\frac{3}{2}}e_3^a e_3^b - \frac{M_1^{ab}}{\sqrt{2}}. \tag{5.26}$$

These are the scalars.

$$M_3^{ab} = \frac{i}{\sqrt{2}}(e_-^a e_-^b - e_+^a e_+^b) \tag{5.27}$$

$$M_4^{ab} = \frac{1}{\sqrt{2}}(e_-^a e_-^b + e_+^a e_+^b) \tag{5.28}$$

are the tensors.

$$M_5^{ab} = i(e_-^{(a}e_3^{b)} - e_+^{(a}e_3^{b)}) \tag{5.29}$$

$$M_6^{ab} = e_-^{(a}e_3^{b)} + e_+^{(a}e_3^{b)} \tag{5.30}$$

are the vectors.

To get M_{ab}^I from M_I^{ab} use the Euclidean metric since one is performing the decomposition in a flat space. In addition to that, all M matrices except for M_1 are trace-free i.e they satisfy:

$$M_I^{ab}e_{ab} = 0 \tag{5.31}$$

for $I = 2...6$.

For the tensor matrices to be transverse they must satisfy:

$$k_a M^{ab} = 0 \tag{5.32}$$

and it can be checked that both M_3^{ab} and M_4^{ab} are transverse. Vector matrices are transverse if:

$$k_a k_b M^{ab} = 0. \tag{5.33}$$

It can be checked that both vector basis are also transverse[6].

5.3.2 Scalar, Vector and Tensor Equations

We decompose the perturbations in this basis as such:

$$\tilde{h}_{ab}(t, \vec{k}) = h_I(t, \vec{k}) M^I_{ab} \tag{5.34}$$

$$\tilde{p}^{ab}(t, \vec{k}) = p^I(t, \vec{k}) M_I^{ab} \tag{5.35}$$

$$\tilde{\xi}^a(t, \vec{k}) = \xi_1(t, \vec{k}) e_1^a + \xi_2(t, \vec{k}) e_2^a + \xi_{||}(t, \vec{k}) e_3^a \tag{5.36}$$

where ξ_1 and ξ_2 are the transverse components of the shift perturbation and $\xi_{||}$ is the parallel part and h_I and p^I are scalar fields associated with different helicity basis. Under this decomposition and definition of orthonormality:

$$\int d^3k [\tilde{p^{ab}} \dot{\tilde{h}}_{ab}] = \int d^3k [p^I \dot{h}_I]. \tag{5.37}$$

[6]These conditions were also discussed in Chapter 3

This defines six pairs of canonically conjugate degrees of freedom (h_I, p^I).

The scalar mode equations are:

$$\dot{h}_1 = -ap_1 - \frac{p}{3a^2}h_1 + \frac{2ia^2}{\sqrt{3}}|k|\xi_{||} \tag{5.38}$$

$$\dot{h}_2 = 2ap_2 + \frac{p}{6a^2}h_2 + 2ia^2\sqrt{\frac{2}{3}}|k|\xi_{||} \tag{5.39}$$

$$\dot{p}_1 = \frac{|k|^2}{3a^3}h_1 - \frac{|k|^2}{3\sqrt{2}a^3}h_2 + \frac{p}{12a^2}p_1 + \frac{p^2}{144a^5}h_1 + \frac{i|k|p}{6\sqrt{3}a}\xi_{||} \tag{5.40}$$

$$\dot{p}_2 = -\frac{|k|^2}{3\sqrt{2}a^3}h_1 + \frac{|k|^2}{6a^3}h_2 - \frac{p}{6a^2}p_2 - \frac{5p^2}{144a^5}h_2 - \frac{i|k|p}{3a}\sqrt{\frac{2}{3}}\xi_{||}. \tag{5.41}$$

The tensor mode equations are:

$$\dot{h}_I = 2ap_I + \frac{p}{6a^2}h_I \tag{5.42}$$

$$\dot{p}_I = -\frac{|k|^2}{2a^3}h_I - \frac{p}{6a^2}p_I - \frac{5p^2}{144a^5}h_I \tag{5.43}$$

where $I = 3, 4$. The vector mode equations are:

$$\dot{h}_5 = 2ap_5 + \frac{p}{6a^2}h_5 + i\sqrt{2}a^2|k|^2\xi_2 \tag{5.44}$$

$$\dot{h}_6 = 2ap_6 + \frac{p}{6a^2}h_6 + i\sqrt{2}a^2|k|^2\xi_1 \tag{5.45}$$

$$\dot{p}_5 = -\frac{p}{6a^2}p_5 - \frac{ip|k|}{3\sqrt{2}a}\xi_2 - \frac{5p^2}{144a^5}h_5 \tag{5.46}$$

$$\dot{p}_6 = -\frac{p}{6a^2}p_6 - \frac{ip|k|}{3\sqrt{2}a}\xi_1 - \frac{5p^2}{144a^5}h_6. \tag{5.47}$$

The longitudinal part of the linearized diffeomorphism constraint is:

$$6a^3(p_1 + \sqrt{2}p_2) + p(h_1 + \sqrt{2}h_2) = 0. \tag{5.48}$$

The transverse parts are:

$$6a^3 p_J + p h_J = 0 \tag{5.49}$$

where $J = 5, 6$. The Maple code used to compute these equations has been provided in Appendix A.4.

5.3.3 Rescaling of Scalar, Vector and Tensor Modes

In standard cosmological perturbation theory, the tensor metric perturbations are defined as:

$$ds^2 = -dt^2 + a^2(e_{ab} + f_{ab})dx^a dx^b \tag{5.50}$$

where $a^2 f_{ab}$ is the perturbation. However, this project uses the line element:

$$ds^2 = -dt^2 + (a^2 e_{ab} + h_{ab})dx^a dx^b. \tag{5.51}$$

We equate the two tensor perturbations:

$$h_{ab} = a^2 f_{ab}. \tag{5.52}$$

We should account for this difference if we are to match results from the two approaches.

1. We do not use $a^2 f_{ab}$ because that introduces unnecessary terms in the second order symplectic term i.e $\int d^3x \dot{h}_{ab} p^{ab}$. This is important because the second order symplectic term is responsible for allowing the decoupling of the SVT canonically conjugate variables.

2. We may also not introduce the re-scaling by multiplying the basis by $a(t)^2$. This is because the basis will then become time-dependent.

3. This re-scaling must be accounted for as such:

$$h_I(t) \to a^2 f_I(t). \tag{5.53}$$

   Therefore, the re-scaling is introduced in the scalar field part of the perturbation.

5.4 Gauge-Fixing and Solving of Constraints

In this section gauges will be fixed and the constraints will be solved. There is a freedom to chose and impose three gauges.

5.4.1 Solving the Transverse Linearized Diffeomorphism Constraint

The gauges picked should be second-class with the linearized diffeomorphism constraint. We pick the gauges:

$$h_J = 0 \tag{5.54}$$

where $J = 5, 6$. These are second-class with equations (5.49). Note that we cannot use the tensor and scalar modes for gauge fixing. This is because these gauge choices are not second-class with the transverse linearized diffeomorphism constraint. Solving equation (5.49) gives:

$$p_J = 0 \tag{5.55}$$

where $J = 5, 6$. We specify that the gauge is preserved under evolution i.e:

$$h_J = \dot{h}_J = 0. \tag{5.56}$$

This leads to $\xi_1 = \xi_2 = 0$ from equations (5.44) till (5.47). Therefore the vector modes and the transverse components of the shift perturbation are zero.

5.4.2 Solving the Longitudinal Linearized Diffeomorphism Constraint

Now one gauge remains to be fixed. Pick that to be:

$$h_2 = 0. \tag{5.57}$$

This is second-class with the longitudinal linearized diffeomorphism constraint in equation (5.48). Note that we cannot use the vector and tensor modes for gauge fixing[7]. This is because these gauge choices are not second-class with the longitudinal linearized diffeomorphism constraint. Solving the constraint gives:

$$p_2 = -\frac{p_1}{\sqrt{2}} - \frac{p}{6\sqrt{2}a^3}h_1. \tag{5.58}$$

We specify that this gauge is preserved in time. Therefore:

$$\dot{h}_2 = 0. \tag{5.59}$$

Substitute the gauge in equation (5.39) for $\dot{h}_2$ and solve for $\xi_{||}$ to get:

$$\xi_{||} = -\frac{i\sqrt{3}}{2a|k|}p_1 - \frac{-ip}{4\sqrt{3}a^4|k|}h_1. \tag{5.60}$$

[7] Given the flow of our calculations the option of using the vector modes for gauge-fixing is not even available. This is because we have already eliminated them in the previous Subsection.

In a decoupled form the scalar mode equation is:

$$\ddot{h}_1 = H^2 h_1. \tag{5.61}$$

Upon the allowed re-scaling, $h_1 = a^2 f_1$, the equation above becomes:

$$\ddot{f}_1 = -4H\dot{f}_1. \tag{5.62}$$

Notice the absence of k in the equation because of which the equation is only dependent on time. It is a useful exercise to reason back from this observation. No k implies that there were no partial derivatives. Lack of partial derivatives implies that there was no notion of connecting different points in space. This leads to the conclusion that the scalar mode is ultralocal: that it is localized fully.

We explore further by substituting for different scale factors. We solve for f_1 and analyze.

1. We start with an exponentially growing universe i.e $a(t) = e^{\lambda t}$ where λ is some constant. The solution is:

$$f_1(t) = C_1 + C_2 e^{-4\lambda t} \tag{5.63}$$

   where C_1 and C_2 are constants of integration. Therefore for an exponential scale factor the scalar mode has a decaying mode and a constant mode.

2. A power-law obeying scale factor $a(t) = t^n$, where n is a positive number (greater than $\frac{1}{4}$) is an interesting case also. In this scenario the scalar mode equation becomes:

$$f_1(t) = D_1 + D_2 t^{-4n+1} \tag{5.64}$$

where D_1 and D_2 are constants of integration. In this case we also get a constant mode and a decaying mode. If we set $n = \frac{2}{3}$ we get the scale factor we found in equation (5.13). In that case we get:

$$f_1(t) = D_1 + D_2 t^{-\frac{5}{3}}. \tag{5.65}$$

5.4.3 The Remaining Degrees of Freedom

Now all the constraints have been solved. The remaining degrees of freedom in phase space are the canonical pairs (h_1, p_1), (h_3, p_3) and (h_4, p_4). We now analyze the latter two. For these modes the equations of motion are (5.42) and (5.43); in decoupled form they are:

$$\ddot{h}_I = -\frac{|k|^2}{a^2} h_I + H\dot{h}_I - H^2 h_I \tag{5.66}$$

where $I = 3, 4$. Upon the allowed rescaling we recover the graviton equation in a the FLRW universe:

$$\ddot{f}_I = -\frac{k^2}{a^2} f_I - 3H\dot{f}_I. \tag{5.67}$$

This matches equation (3.70) which was obtained for helicity tensors via the standard approach.

5.5 Summary

Chapter 4 was ended after obtaining the machinery that will allow one to compute for the spatial and temporal behavior of any space-time. This chapter was started by specifying the three-metric, three-momentum and shift for cosmology; these quantities were prescribed with backgrounds and perturbations. The evolution equations and the diffeomorphism constraints were derived to linear order. A spatial Fourier transform was performed on the results and a basis was specified that decomposed these results into their SVT components. This basis was orthonormal and did not depend on time and the Fourier mode. Following that gauges were picked and the first-class constraints were solved. As predicted in the previous chapter 6 degrees of freedom remained: 4 in the tensor modes and 2 in the scalar modes.

We found an ultralocal scalar mode. Under two different specifications for the scale factor it was seen that the scalar mode had a constant term and a decaying mode. The vector modes vanished. Finally the tensor modes gave the graviton equations.

Chapter 6

Summary and Conclusions

We now give a brief summary of the main results from each section.

In Chapter 2 we used perturbation theory in GR to study weak gravitational fields in Minkowski spacetime. We found that we have gauge freedom in the theory and explored two different gauge choices to solve the Einstein equations. Both routes gave a similar result: two degrees of freedom in the metric perturbation that propagated as waves. These two propagating degrees of freedom were in the traceless, transverse and spatial part of the metric perturbations and they represented gravitational waves.

In Chapter 3 we studied perturbations in the FLRW universe. We found the corresponding propagating degrees of freedom from the traceless and transverse part of the tensor perturbation. It was also seen that the vector perturbations in cosmology vanish and that the scalar perturbations exhibit non-trivial dynamics.

In Chapter 4 we used the Hamiltonian formulation to study scalar fields, GR and a system of GR and dust. We found that the dynamics of a scalar field in a Minkowski spacetime were identical to the gravitational wave equation in flat spacetime. We found a similar relationship between the propagation behavior of a scalar field and gravitational waves in the homogeneous and isotropic universe. This is meaningful because we can represent the propagating gravity degrees of freedom in the metric using scalar fields. We next studied GR in its Hamiltonian form and found that the Hamiltonian in GR vanishes. We then considered a system of GR and a pressure-less, timelike, massive dust in its Hamiltonian form. We picked the gauge that dust is time and solved the Hamiltonian constraint to get a physical Hamiltonian. This system can be used to study perturbation theory in GR.

In Chapter 5 we studied cosmological perturbation theory in this Hamiltonian framework. We found an ultralocal scalar mode. Next we considered the scale factors for a universe that is exponentially growing and a universe that expands according to the power law. It was found that in both cases the scalar mode had a decaying part and a constant part, the latter which we could set to zero. It was checked that the vector modes vanish. Lastly we found that the traceless, transverse tensor perturbation equations were the gravitational wave equation.

There are two purposes of studying cosmological perturbation theory in this new framework:

1. The dust-time model is used for research in quantum gravity. Any

framework that operates in this regime should produce the known results in the classical realm. Cosmology is a good test-bed because the result we expect (of the propagation behavior of gravitational waves) is known and also because the symmetries of homogeneity and isotropy do not make the calculations overly complex.

2. There is the intrigue of what happens to the scalar mode once we use the dust field to fix the time gauge. The scalar degrees of freedom consequently reside in the three-metric and its conjugate momentum. We find that when we use the model to study cosmological perturbation theory, these scalar modes manifest themselves as ultralocal degrees of freedom that are not important at late times.

To ask what research can be done next, it is enlightening to review that cosmological perturbation theory allows us to track how small perturbations in the universe evolve in space and time. We can next explore how these perturbations relate to the anisotropies of the CMB. We could also question the role these fluctuations played in the formation of galaxies. To answer this we will have to add to our model a scalar field that represents matter. Then we will be able to study interactions between the gravitational and matter degrees of freedom.

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Appendix A

Appendix

A.1 Second-Class Constraints

The first-class constraints are of interest and second-class constraints have to be removed from the theory. To see why, a small foray into quantization is required. To quantize a theory:

1. Take the case where all the constraints are first-class.
2. Promote the canonical variables to operators. We recall that the basic Poisson Bracket in the classical theory is:

$$[x, p]_{PB} = 1. \tag{A.1}$$

This Poisson Bracket from the classical theory should be promoted to the commutator relation in quantum theory:

$$[x, p] = i \tag{A.2}$$

where it is understood that the position and momentum are now operators.

3. To find how the system evolves in time we use the time-dependent Schrödinger equation:

$$i\hbar \frac{d\psi}{dt} = H\psi \tag{A.3}$$

where ψ is the state we are examining and H is the Hamiltonian.

4. Impose supplementary conditions on ψ as such:

$$\phi_j \psi = 0. \tag{A.4}$$

It is a useful exercise to examine the consistency of the last assumption more closely.

$$\begin{aligned} \phi_{j'} \phi_j \psi &= 0 \\ \phi_j \phi_{j'} \psi &= 0. \end{aligned} \tag{A.5}$$

Subtracting one from the other gives:

$$\begin{aligned} [\phi_{j'} \phi_j - \phi_{j'} \phi_j] \psi &= 0 \\ [\phi_{j'}, \phi_j] \psi &= 0 \end{aligned} \tag{A.6}$$

where $[\phi_{j'}, \phi_j]$ is the commutator. This second condition on ψ is necessary for consistency. The commutator should satisfy the relation:

$$[\phi_j, \phi_{j'}] = c_{jj'j''}\phi_{j''} \tag{A.7}$$

where $c_{jj'j''}$ is a constant. But this corresponds to the Poisson Bracket relation for first-class constraints from the classical theory. So really one has just the initial supplementary condition in equation (A.4) and the relation above naturally follows from the definition of first-class constraints.

This is where the second-class constraints are problematic. As a particular example assume one has two second-class constraints $q_1 \approx 0$ and $p_1 \approx 0$ in the classical theory. Recall in classical theory the Poisson Bracket of two second-class constraints is non-zero and therefore we cannot write the right hand side as a linear combination of constraints. Their commutator follows a similar relation and because of this we cannot write:

$$[q_1, p_1]\psi = 0 \tag{A.8}$$

in the quantum theory. This is how a second-class constraint ruins the consistency for the supplementary conditions in quantum theory. Therefore they are eliminated and the theory is expressed in terms of the remaining first-class constraints.

To eliminate these second-class constraints we could take q_1 and p_1 to be identically zero and set up the theory in terms of the degrees of freedom with

$n = 2, 3...N$. The Poisson Bracket for two functions $f(q_n, p_n)$ and $g(q_n, p_n)$ is defined as:

$$[f, g]_{PB} = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n}. \tag{A.9}$$

for $n = 2, 3...N$. We now consider a more general case where the second-class constraints in the theory are $p_1 \approx 0$ and $q_1 \approx f(q_r, p_r)$ where $r = 2, 3...N$. In this case we could drop out the number 1 degree of freedom if we substitute $f(q_r, p_r)$ for q_1 in the Hamiltonian and all other constraints [3]. This allows us to remove the second-class constraints and set up the quantum theory in terms of the other degrees of freedom.

A.2 The ADM Foliation of Spacetime

ADM stands for Arnowitt, Deser and Misner. They postulated a separation of spacetime into leaves of space and a parameter time. This separation of space from time, which is also referred to as foliation, facilitated the Hamiltonian formulation of General Relativity. The ADM foliation of spacetime is also used to derive the Hamiltonian formulation of other fields on a curved background. In this Appendix section we discuss the ADM foliation and how we can do calculus on the surface of constant t [10] [15]:

1. We separate spacetime into surfaces Σ and a parameter t that is real. We call t a time function and impose that $t = constant$ gives the surfaces Σ_t. These surfaces of constant t are assumed to be spacelike.

We also impose that there exists a smooth map with a smooth inverse between all Σ_t.

2. To introduce evolution we define a timelike vector field t^a that satisfies $t^a(\nabla t)_a = 1$ and defines the same point in space at a different instants of time. Thus evolution is given by the Lie derivative with respect to t^a. For example the time derivative of a symmetric tensor p_{ab} is:

$$\begin{aligned} \dot{p}_{ab} &= \pounds_t p_{ab} \\ &= t^c p_{ab;c} + 2p_{c(a}t^c_{;b)}. \end{aligned} \tag{A.10}$$

3. We then introduce a unit timelike covariant normal to the spacelike surface which we call n_a. Its contravariant form is given by $n^a = g^{ab}n_b$. Since n^a is a unit timelike vector

$$n^a n_a = -1. \tag{A.11}$$

4. We define the projection operator to the tangent space of the spacelike surface Σ_t as:

$$q^a_b = \delta^a_b + n^a n_b \tag{A.12}$$

where δ^a_b is the Kronecker Delta. We may view tensor P^a_b to be on the tangent space to Σ_t if P^a_b satisfies the following equation:

$$P^a_b = q^a_c q^d_b P^c_d. \tag{A.13}$$

5. By projecting the full metric g_{ab} onto Σ_t we get the metric induced on Σ_t; we call that q_{ab}.

$$\begin{aligned} q_{ab} =& q_a^c q_b^d g_{cd} \\ =& g_{ab} + n_a n_b. \end{aligned} \tag{A.14}$$

6. We prescribe the covariant derivative on Σ_t. We call it D_a and require that:

   (a) It should take the covariant derivatives along all directions except for that of the normal. This is ensured by projecting the full covariant derivative on Σ_t as $q_c^a \nabla_a$.

   (b) The resultant quantity of this covariant differentiation should lie on the tangent space to Σ_t. Therefore the covariant derivative of a spacetime tensor Q_b^a on Σ_t should give:

   $$D_c Q_b^a = q_u^a q_b^v q_c^f \nabla_f Q_v^u \tag{A.15}$$

   where it can be checked that the resultant quantity is on the tangent plane to Σ_t.

   (c) D_a should be compatible with q_{ab} i.e:

   $$D_a q_{bc} = 0. \tag{A.16}$$

(d) It should obey linearity and Leibnitz rule i.e for any two abstract tensors A and B it should respectively satisfy:

$$\begin{aligned} D(A+B) &= DA + DB \\ D(AB) &= (DA)B + A(DB). \end{aligned} \tag{A.17}$$

7. We next re-express t^a in terms of its components tangential and normal to Σ_t. Define the lapse as the projection of t^a along the normal:

$$N = g_{ab} t^a n^b \tag{A.18}$$

and the shift, N^a, as the projection of t^a on Σ_t:

$$N^a = q^a_b t^b. \tag{A.19}$$

We rewrite t^a as:

$$t^a = N n^a + N^a. \tag{A.20}$$

Therefore (N, N^a) are also indicative of a basis in this structure.

A.3 Equations of Motion for the Perturbations

One has the perturbed three-metric and the three-momentum:

$$
\begin{aligned}
q_{ab}(t,\vec{x}) &= a^2(t)e_{ab} + \epsilon h_{ab}(t,\vec{x}) \\
\pi^{ab}(t,\vec{x}) &= \frac{p(t)}{6a(t)}e^{ab} + \epsilon p^{ab}(t,\vec{x}).
\end{aligned}
\tag{A.21}
$$

The determinant is:

$$q = a^6. \tag{A.22}$$

Postulate the inverse metric as such:

$$q^{ab} = Xe^{ab} + \epsilon Y^{ab}. \tag{A.23}$$

The metric and its inverse satisfy the following relation:

$$q_{ab}q^{bc} = \delta^c_a. \tag{A.24}$$

Substitute (A.23) in the equation above and ignore all terms of second order:

$$
\begin{aligned}
(a^2 e_{ab} + \epsilon h_{ab})(Xe^{bc} + \epsilon Y^{bc}) &= \delta^c_a \\
a^2 X \delta^a_c + \epsilon(a^2 e_{ab} Y^{bc} + h_{ab} X e^{bc}) &= \delta^c_a \\
\delta^c_a(a^2 X - 1) + \epsilon(a^2 e_{ab} Y^{bc} + h_{ab} X e^{bc}) &= 0.
\end{aligned}
\tag{A.25}
$$

In the last line we have brought δ_a^c on the left hand side. The zeroth order is satisfied if:

$$X = \frac{1}{a^2} \tag{A.26}$$

and that leaves

$$a^2 e_{ab} Y^{bc} + \frac{h_{ab} e^{bc}}{a^2} = 0. \tag{A.27}$$

Solving for Y^{ab} gives:

$$Y^{ab} = -\frac{e^{ac} e^{bd}}{a^4} h_{cd}. \tag{A.28}$$

The inverse-metric thus becomes:

$$q^{ab} = \frac{e^{ab}}{a^2} - \epsilon \frac{e^{ac} e^{bd}}{a^4} h_{cd}. \tag{A.29}$$

For convenience π_b^a is computed next. From here computing the trace of the momentum and π_{ab} does not take much effort:

$$\begin{aligned} \pi_b^a &= \pi^{ac} q_{cb} \\ &= (\frac{p}{6a} e^{ac} + \epsilon p^{ac})(a^2 e_{cb} + \epsilon h_{cb}) \\ &= \frac{pa}{6} \delta_b^a + \epsilon(a^2 e_{cb} p^{ac} + \frac{p}{6a} e^{ac} h_{cb}). \end{aligned} \tag{A.30}$$

To find the trace of the momentum make $b = a$ in the equation above:

$$\pi = \frac{pa}{2} + \epsilon(a^2 e_{ac} p^{ac} + \frac{p}{6a} e^{ac} h_{ac}). \tag{A.31}$$

To calculate π_{ab} lower one index on π_b^a; this corresponds to multiplying with the three-metric:

$$\begin{aligned}\pi_{ab} &= \pi_b^c q_{ac} \\ &= [\frac{pa}{6}\delta_b^c + \epsilon(a^2 e_{db} p^{cd} + \frac{p}{6a} e^{cd} h_{db})][a^2 e_{ac} + \epsilon h_{ac}] \\ &= \frac{pa^3}{6} e_{ab} + \epsilon(a^4 e_{ac} e_{bd} p^{cd} + \frac{pa}{3} h_{ab}).\end{aligned} \tag{A.32}$$

The curvature terms are dealt with next and one will start with the Christoffel symbols:

$$\begin{aligned}\Gamma_{abc} &= \frac{1}{2}(q_{ab,c} + q_{ca,b} - q_{cb,a}) \\ &= \frac{\epsilon}{2}(h_{ab,c} + h_{ca,b} - h_{cb,a}).\end{aligned} \tag{A.33}$$

One should at this point, recall that the partial derivatives are spatial because the metric itself is on the spatial slice; hence the zeroth order term of the metric will give zero when the partial derivative acts on it. The Christoffel symbols of second kind can thus be written as:

$$\Gamma^a_{bc} = \frac{\epsilon e^{ad}}{2a^2}(h_{db,c} + h_{cd,b} - h_{cb,d}). \tag{A.34}$$

Note that the Christoffel is already of order ϵ. Therefore terms in curvature involving products of Christoffels will not exist in linear theory.

By extension, the definition for the Ricci tensor becomes:

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ca}. \tag{A.35}$$

This reduces down to:

$$R_{ab} = \frac{\epsilon e^{cd}}{2a^2}(h_{bd,ac} + h_{ac,bd} - h_{ab,cd} - h_{cd,ab}). \tag{A.36}$$

To find the Ricci scalar with both indices up multiply twice by inverse-metric as such:

$$\begin{aligned} R^{ab} &= q^{ac}q^{bd}R_{cd} \\ &= \frac{\epsilon e^{ac}e^{bd}e^{ij}}{2a^6}(h_{dj,ci} + h_{ci,dj} - h_{cd,ij} - h_{ij,cd}). \end{aligned} \tag{A.37}$$

The final piece to compute is the three-Ricci scalar which can be found as such:

$$\begin{aligned} R &= q^{cd}R_{cd} \\ &= \frac{\epsilon e^{cd}e^{ij}}{2a^4}(h_{dj,ci} + h_{ci,dj} - h_{cd,ij} - h_{ij,cd}). \end{aligned} \tag{A.38}$$

Recall the Einstein field equations in the dust-time gauge. Start with the one for the three-metric and substitute for the different terms we just calculated:

$$\begin{aligned} \dot{q}_{ab} &= \frac{2}{\sqrt{q}}[\pi_{ab} - \frac{\pi q_{ab}}{2}] + 2D_{(a}N_{b)} \\ 2a\dot{a}e_{ab} + \epsilon\dot{h}_{ab} &= -\frac{p}{6}e_{ab} \\ &+ \epsilon(2ae_{ac}e_{bd}p^{cd} - ap^{cd}e_{cd}e_{ab} + \frac{p}{6a^2}h_{ab} - \frac{p}{6a^2}e^{cd}h_{cd}e_{ab} + 2a^2 e_{c(a}\xi^c_{,b)}). \end{aligned} \tag{A.39}$$

The zeroth order equation of motion can be re-arranged to give:

$$\dot{a} = -\frac{p}{12a}. \tag{A.40}$$

The equation of motion for the perturbation to linear order is:

$$\dot{h}_{ab} = 2ae_{ac}e_{bd}p^{cd} - ap^{cd}e_{cd}e_{ab} + \frac{p}{6a^2}h_{ab} - \frac{p}{6a^2}e^{cd}h_{cd}e_{ab} + 2a^2e_{c(a}\xi^c_{,b)}. \tag{A.41}$$

Now for the equation of motion for the momentum.

$$\begin{aligned}\dot{\pi}^{ab} =& -\sqrt{q}(R^{ab} - \frac{{}^3Rq^{ab}}{2}) + \frac{q^{ab}}{2\sqrt{q}}(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}) - \frac{2}{\sqrt{q}}(\pi^a_c\pi^{bc} - \frac{\pi\pi^{ab}}{2})\\ &+ \sqrt{q}D_c(\frac{\pi^{ab}N^c}{\sqrt{q}}) - 2\pi^{c(a}D_cN^{b)}\\ (\frac{\dot{p}}{6a} + \frac{p^2}{72a^3})e^{ab} + \epsilon\dot{p}^{ab} =& \frac{p^2}{144a^3}e^{ab}\\ &+ \epsilon[-\frac{p}{6a^2}p^{ab} + \frac{p}{12a^2}e_{cd}p^{cd}e^{ab} - \frac{5p^2}{144a^5}e^{ac}e^{bd}h_{cd}\\ &+ \frac{p^2}{72a^5}e^{cd}h_{cd}e^{ab} - \frac{p}{3a}e^{c(a}\xi^{b)}_{,c} + \frac{p}{6a}e^{ab}\xi^c_{,c}\\ &+ \frac{e^{ij}}{2a^3}(\frac{e^{ab}e^{cd}}{2} - e^{ac}e^{bd})(h_{dj,ci} + h_{ci,dj} - h_{cd,ij} - h_{ij,cd})].\end{aligned} \tag{A.42}$$

The zeroth order equation of motion can be rearranged to give:

$$\dot{p} = -\frac{p^2}{24a^2}. \tag{A.43}$$

The equation of motion for the perturbation to linear order is:

$$\dot{p}^{ab} = -\frac{p}{6a^2}p^{ab} + \frac{p}{12a^2}e_{cd}p^{cd}e^{ab} - \frac{5p^2}{144a^5}e^{ac}e^{bd}h_{cd} + \frac{p^2}{72a^5}e^{cd}h_{cd}e^{ab} - \frac{p}{3a}e^{c(a}\xi^{b)}_{,c} + \frac{p}{6a}e^{ab}\xi^{c}_{,c}$$
$$+ \frac{e^{ij}}{2a^3}(\frac{e^{ab}e^{cd}}{2} - e^{ac}e^{bd})(h_{dj,ci} + h_{ci,dj} - h_{cd,ij} - h_{ij,cd}). \tag{A.44}$$

Finally, the diffeormorphism constraint is:

$$D_b(\sqrt{q}\pi^{ab}) = 0$$
$$p^{ab}_{,b} + \frac{p}{12a^3}(e^{ad}e^{bc} + e^{ac}e^{bd})(h_{db,c} + h_{cd,b} - h_{cb,d}) = 0. \tag{A.45}$$

A.4 Maple Code for the Scalar, Vector, Tensor Decomposition

```
> restart;
> with(Physics):
  with(LinearAlgebra):
  Setup(mathematicalnotation = true):
  Coordinates(X):
  Coordinates(X = cartesian):
  Setup(dimension=[3,`+`]):
```

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (x1, x2, x3, x4)\}$

Systems of spacetime Coordinates are: $\{X = (x1, x2, x3, x4)\}$

Default differentiation variables for d_, D_ and dAlembertian are: $\{X = (x, y, z, t)\}$

Systems of spacetime Coordinates are: $\{X = (x, y, z, t)\}$

Changing the signature of the tensor spacetime to: $+ + +$

The dimension and signature of the tensor space are set to: $[3, + + +]$

Systems of spacetime Coordinates are: $\{X = (x, y, t)\}$ **(1)**

Basic Ingredients

```
> g_[]; #Sets up flat metric for these calculations
  e1:=<1,0,0>: #Establishes e1 to be along the x-axis.
  e2:=<0,1,0>:
  e3:=<0,0,1>: #This is the k unit vector
  e_p:= (e1+I*e2)/sqrt(2): #This is the eigenvector - "e plus" -
  of the rotation matrix.
  e_m:= (e1-I*e2)/sqrt(2):
  Define(e_1[~mu]=e1); #This defines all vectors with upper
  indices
  Define(e_2[~mu]=e2);
  Define(e_3[~mu]=e3);
  Define(k[~mu]=e_3[~mu]*k); #This is the full k-vector
```

$$g_{\mu,\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Defined objects with tensor properties

$$\{\gamma_\mu, \sigma_\mu, X_\mu, \partial_\mu, e\_1^\mu, g_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\}$$

Defined objects with tensor properties

$$\{\gamma_\mu, \sigma_\mu, X_\mu, \partial_\mu, e\_1^\mu, e\_2^\mu, g_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\}$$

Defined objects with tensor properties

$$\{\gamma_\mu, \sigma_\mu, X_\mu, \partial_\mu, e\_1^\mu, e\_2^\mu, e\_3^\mu, g_{\mu,\nu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\}$$

Defined objects with tensor properties

$$\{\gamma_\mu, \sigma_\mu, X_\mu, \partial_\mu, e\_1^\mu, e\_2^\mu, e\_3^\mu, g_{\mu,\nu}, k^\mu, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\} \quad \textbf{(1.1)}$$

M-Matrices

```
> M1:= 1/sqrt(3)* (OuterProductMatrix(e_p,e_m) +
OuterProductMatrix(e_m,e_p)  + OuterProductMatrix(e3,e3)):
Define(M_1[~mu,~nu] = M1): #This part of the code sets up the M matrix with upper indices, and displays the same M matrix with lower indices; one should notice that they are identical. One may check that the prescription is the same as the one given in equations 5.25-5.30.
M_1[~mu,~nu,Matrix];
M_1[mu,nu,Matrix];
```

Defined objects with tensor properties

$$M\_1^{\mu,\nu} = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$$

$$M\_1_{\mu,\nu} = \begin{bmatrix} \frac{\sqrt{3}}{3} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & \frac{\sqrt{3}}{3} \end{bmatrix} \quad \textbf{(2.1)}$$

```
> M2:= sqrt(3)/sqrt(2)*OuterProductMatrix(e3,e3) - M1/sqrt(2):
Define(M_2[~mu,~nu] = M2);
M_2[~mu,~nu,Matrix];
M_2[mu,nu,Matrix];
```

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, M\_1^{\mu,\nu}, M\_2^{\mu,\nu}, \sigma_{\mu}, X_{\mu}, \partial_{\mu}, e\_1^{\mu}, e\_2^{\mu}, e\_3^{\mu}, g_{\mu,\nu}, k^{\mu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\right\}$$

$$M\_2^{\mu,\nu} = \begin{bmatrix} -\frac{\sqrt{3}\sqrt{2}}{6} & 0 & 0 \\ 0 & -\frac{\sqrt{3}\sqrt{2}}{6} & 0 \\ 0 & 0 & \frac{\sqrt{3}\sqrt{2}}{3} \end{bmatrix}$$

$$M\_2_{\mu,\nu} = \begin{bmatrix} -\frac{\sqrt{3}\sqrt{2}}{6} & 0 & 0 \\ 0 & -\frac{\sqrt{3}\sqrt{2}}{6} & 0 \\ 0 & 0 & \frac{\sqrt{3}\sqrt{2}}{3} \end{bmatrix} \quad (2.2)$$

```
> M3:=I/sqrt(2)*(OuterProductMatrix(e_m,e_m) - OuterProductMatrix
  (e_p,e_p)):
  Define(M_3[~mu,~nu] = M3);
  M_3[~mu,~nu,Matrix];
  M_3[mu,nu,Matrix];
```

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, M\_1^{\mu,\nu}, M\_2^{\mu,\nu}, M\_3^{\mu,\nu}, \sigma_{\mu}, X_{\mu}, \partial_{\mu}, e\_1^{\mu}, e\_2^{\mu}, e\_3^{\mu}, g_{\mu,\nu}, k^{\mu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\right\}$$

$$M\_3^{\mu,\nu} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M\_3_{\mu,\nu} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.3)$$

```
> M4:=1/sqrt(2)*(OuterProductMatrix(e_m,e_m) + OuterProductMatrix
  (e_p,e_p)):
  Define(M_4[~mu,~nu] = M4);
  M_4[~mu,~nu,Matrix];
  M_4[mu,nu,Matrix];
```

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, M\_1^{\mu,\nu}, M\_2^{\mu,\nu}, M\_3^{\mu,\nu}, M\_4^{\mu,\nu}, \sigma_{\mu}, X_{\mu}, \partial_{\mu}, e\_1^{\mu}, e\_2^{\mu}, e\_3^{\mu}, g_{\mu,\nu}, k^{\mu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\right\}$$

$$M\_4^{\mu,\nu} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M\_4_{\mu,\nu} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.4)$$

```
> M5:= I/2*(OuterProductMatrix(e_m,e3) + OuterProductMatrix(e3,
  e_m) - OuterProductMatrix(e_p,e3) - OuterProductMatrix(e3,e_p))
  :
  Define(M_5[~mu,~nu] = M5);
  M_5[~mu,~nu,Matrix];
  M_5[mu,nu,Matrix];
```

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, M\_1^{\mu,\nu}, M\_2^{\mu,\nu}, M\_3^{\mu,\nu}, M\_4^{\mu,\nu}, M\_5^{\mu,\nu}, \sigma_{\mu}, X_{\mu}, \partial_{\mu}, e\_1^{\mu}, e\_2^{\mu}, e\_3^{\mu}, g_{\mu,\nu}, k^{\mu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\right\}$$

$$M\_5^{\mu,\nu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$M\_5_{\mu,\nu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \quad (2.5)$$

```
> M6:= 1/2*(OuterProductMatrix(e_m,e3) + OuterProductMatrix(e3,
  e_m) + OuterProductMatrix(e_p,e3) + OuterProductMatrix(e3,e_p))
  ;
  Define(M_6[~mu,~nu] = M6);
  M_6[~mu,~nu,Matrix];
  M_6[mu,nu,Matrix];
```

$$M6 := \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}$$

Defined objects with tensor properties

$$\left\{\gamma_{\mu}, M\_1^{\mu,\nu}, M\_2^{\mu,\nu}, M\_3^{\mu,\nu}, M\_4^{\mu,\nu}, M\_5^{\mu,\nu}, M\_6^{\mu,\nu}, \sigma_{\mu}, X_{\mu}, \partial_{\mu}, e\_1^{\mu}, e\_2^{\mu}, e\_3^{\mu}, g_{\mu,\nu},\right.$$

$k^{\mu}, \delta_{\mu,\nu}, \epsilon_{\alpha,\mu,\nu}\}$

$$M\_6^{\mu,\nu} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}$$

$$M\_6_{\mu,\nu} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \quad (2.6)$$

```
> #The next part of the code checks orthonormality of the M
matrices. The SumOverRepeatedIndices command takes the trace of
the matrices being multiplied. This is consistent with the
definition of orthonormality.
```

```
> SumOverRepeatedIndices(M_1[~mu,~nu].M_1[mu,nu]);
SumOverRepeatedIndices(M_1[~mu,~nu].M_2[mu,nu]);
SumOverRepeatedIndices(M_1[~mu,~nu].M_3[mu,nu]);
SumOverRepeatedIndices(M_1[~mu,~nu].M_4[mu,nu]);
SumOverRepeatedIndices(M_1[~mu,~nu].M_5[mu,nu]);
SumOverRepeatedIndices(M_1[~mu,~nu].M_6[mu,nu]);
```

$$1$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0 \quad (2.7)$$

```
> SumOverRepeatedIndices(M_2[~mu,~nu].M_1[mu,nu]);
SumOverRepeatedIndices(M_2[~mu,~nu].M_2[mu,nu]);
SumOverRepeatedIndices(M_2[~mu,~nu].M_3[mu,nu]);
SumOverRepeatedIndices(M_2[~mu,~nu].M_4[mu,nu]);
SumOverRepeatedIndices(M_2[~mu,~nu].M_5[mu,nu]);
SumOverRepeatedIndices(M_2[~mu,~nu].M_6[mu,nu]);
```

$$0$$
$$1$$
$$0$$
$$0$$
$$0$$
$$0 \quad (2.8)$$

```
> SumOverRepeatedIndices(M_3[~mu,~nu].M_1[mu,nu]);
SumOverRepeatedIndices(M_3[~mu,~nu].M_2[mu,nu]);
SumOverRepeatedIndices(M_3[~mu,~nu].M_3[mu,nu]);
SumOverRepeatedIndices(M_3[~mu,~nu].M_4[mu,nu]);
```

```
SumOverRepeatedIndices(M_3[~mu,~nu].M_5[mu,nu]);
SumOverRepeatedIndices(M_3[~mu,~nu].M_6[mu,nu]);
```

$$0$$
$$0$$
$$1$$
$$0$$
$$0$$
$$0 \tag{2.9}$$

```
> SumOverRepeatedIndices(M_4[~mu,~nu].M_1[mu,nu]);
SumOverRepeatedIndices(M_4[~mu,~nu].M_2[mu,nu]);
SumOverRepeatedIndices(M_4[~mu,~nu].M_3[mu,nu]);
SumOverRepeatedIndices(M_4[~mu,~nu].M_4[mu,nu]);
SumOverRepeatedIndices(M_4[~mu,~nu].M_5[mu,nu]);
SumOverRepeatedIndices(M_4[~mu,~nu].M_6[mu,nu]);
```

$$0$$
$$0$$
$$0$$
$$1$$
$$0$$
$$0 \tag{2.10}$$

```
> SumOverRepeatedIndices(M_5[~mu,~nu].M_1[mu,nu]);
SumOverRepeatedIndices(M_5[~mu,~nu].M_2[mu,nu]);
SumOverRepeatedIndices(M_5[~mu,~nu].M_3[mu,nu]);
SumOverRepeatedIndices(M_5[~mu,~nu].M_4[mu,nu]);
SumOverRepeatedIndices(M_5[~mu,~nu].M_5[mu,nu]);
SumOverRepeatedIndices(M_5[~mu,~nu].M_6[mu,nu]);
```

$$0$$
$$0$$
$$0$$
$$0$$
$$1$$
$$0 \tag{2.11}$$

```
> SumOverRepeatedIndices(M_6[~mu,~nu].M_1[mu,nu]);
SumOverRepeatedIndices(M_6[~mu,~nu].M_2[mu,nu]);
SumOverRepeatedIndices(M_6[~mu,~nu].M_3[mu,nu]);
SumOverRepeatedIndices(M_6[~mu,~nu].M_4[mu,nu]);
SumOverRepeatedIndices(M_6[~mu,~nu].M_5[mu,nu]);
SumOverRepeatedIndices(M_6[~mu,~nu].M_6[mu,nu]);
```

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$1 \tag{2.12}$$

Results from the Cosmological Case

```
> #This code defines the scalar, vector and tensor decomposition of the perturbations and their time derivatives. This part of the code also decomposes the perturbation to the shift. One may check that the prescription is the same as the one given in equations 5.34-5.36.

Define(dot_h[mu,nu]= dot_h1*M_1[mu,nu] + dot_h2*M_2[mu,nu] + dot_h3*M_3[mu,nu] +dot_h4*M_4[mu,nu] + dot_h5*M_5[mu,nu] + dot_h6*M_6[mu,nu]):

Define(h[mu,nu]= h1*M_1[mu,nu] + h2*M_2[mu,nu] + h3*M_3[mu,nu] + h4*M_4[mu,nu] + h5*M_5[mu,nu] + h6*M_6[mu,nu]):

Define(dot_p[~mu,~nu]= dot_p1*M_1[~mu,~nu] + dot_p2*M_2[~mu,~nu] + dot_p3*M_3[~mu,~nu] +dot_p4*M_4[~mu,~nu] + dot_p5*M_5[~mu,~nu] + dot_p6*M_6[~mu,~nu]):

Define(p[~mu,~nu]= p1*M_1[~mu,~nu] + p2*M_2[~mu,~nu] + p3*M_3[~mu,~nu] + p4*M_4[~mu,~nu] + p5*M_5[~mu,~nu] + p6*M_6[~mu,~nu]):

Define(xi[~mu]=xi_1*e_1[~mu] + xi_2*e_2[~mu] + xi_p*e_3[~mu]):

Define(e[~mu,~nu]=g_[~mu,~nu]):
```

Defined objects with tensor properties

Defined objects with tensor properties

Defined objects with tensor properties

Defined objects with tensor properties

Defined objects with tensor properties

Defined objects with tensor properties **(3.1)**

```
> #This is the spatial Fourier transform of the metric perturbation equation of motion. This will be decomposed along scalar, vector and tensor in the next command. This is the same as equation 5.21.

metric_EOM_c:= dot_h[mu,nu]= 2*a*SumOverRepeatedIndices(e[mu,alpha].e[nu,beta].p[~alpha,~beta]) - p/(6*a^2)*SumOverRepeatedIndices(e[~alpha,~beta].h[alpha,beta])*e[mu,nu] - a*SumOverRepeatedIndices(p[~alpha,~beta].e[alpha,beta])*e[mu,nu] + p/(6*a^2)*h[mu,nu] + a^2*I*(SumOverRepeatedIndices(xi[~alpha].e[alpha,mu].k*e_3[nu])+SumOverRepeatedIndices(xi[~alpha].e[alpha,nu].k*e_3[mu])):
```

```
> decomposed_metric_EOM_c[1]:= SumOverRepeatedIndices(lhs(metric_EOM_c).M_1[~mu,~nu]) = simplify(SumOverRepeatedIndices(rhs(metric_EOM_c).M_1[~mu,~nu])): #The left hand side of the equation will pick h1_dot and the right hand side will pick out the projection of the metric perturbation along M1 and simplify

decomposed_metric_EOM_c[2]:= SumOverRepeatedIndices(lhs(metric_EOM_c).M_2[~mu,~nu]) = simplify(SumOverRepeatedIndices(rhs(metric_EOM_c).M_2[~mu,~nu])):
```

```
decomposed_metric_EOM_c[3]:= SumOverRepeatedIndices(lhs
(metric_EOM_c).M_3[~mu,~nu]) = simplify(SumOverRepeatedIndices
(rhs(metric_EOM_c).M_3[~mu,~nu])):
decomposed_metric_EOM_c[4]:= SumOverRepeatedIndices(lhs
(metric_EOM_c).M_4[~mu,~nu]) = simplify(SumOverRepeatedIndices
(rhs(metric_EOM_c).M_4[~mu,~nu])):
decomposed_metric_EOM_c[5]:= SumOverRepeatedIndices(lhs
(metric_EOM_c).M_5[~mu,~nu]) = simplify(SumOverRepeatedIndices
(rhs(metric_EOM_c).M_5[~mu,~nu])):
decomposed_metric_EOM_c[6]:= SumOverRepeatedIndices(lhs
(metric_EOM_c).M_6[~mu,~nu]) = simplify(SumOverRepeatedIndices
(rhs(metric_EOM_c).M_6[~mu,~nu])):
```

```
> #This is the spatial Fourier transform of the momentum
perturbation equation of motion. This is the same as equation
5.22. This will be decomposed along scalar, vector and tensor
in the next command.

p_EOM_c:= dot_p[~mu,~nu] =  - p/(6*a^2)*p[~mu,~nu] + p/(12*a^2)
*SumOverRepeatedIndices(p[~alpha,~beta].e[alpha,beta])*e[~mu,
~nu] - (I*k*p)/(6*a)*(SumOverRepeatedIndices(e[~mu,~alpha].e_3
[alpha].xi[~nu]) + SumOverRepeatedIndices(e[~nu,~alpha].e_3
[alpha].xi[~mu])) + (I*p*k)/(6*a)*e[~mu,~nu].xi[~alpha].e_3
[alpha] + (p^2)/(72*a^5)*SumOverRepeatedIndices(h[alpha,beta].e
[~alpha,~beta])*e[~mu,~nu] - 5*p^2/(144*a^5)*
SumOverRepeatedIndices(e[~mu,~alpha].e[~nu,~beta].h[alpha,beta]
) - k^2/(2*a^3)*SumOverRepeatedIndices(e[~gamma,~sigma].(e_3
[alpha].e_3[gamma].h[sigma,beta] + e_3[beta].e_3[sigma].h
[gamma,alpha] - e_3[gamma].e_3[sigma].h[alpha,beta] - e_3
[alpha].e_3[beta].h[gamma,sigma] ).(e[~mu,~nu].e[~alpha,~beta]
/2 - e[~mu,~alpha].e[~nu,~beta])):
```

```
> decomposed_p_EOM_c[1]:= SumOverRepeatedIndices(lhs(p_EOM_c).M_1
[mu,nu]) = simplify(SumOverRepeatedIndices(rhs(p_EOM_c).M_1[mu,
nu])): #The left hand side of the equation will pick p1_dot and
the right hand side will pick out the projection of the
momentum perturbation along M1 and simplify

decomposed_p_EOM_c[2]:= SumOverRepeatedIndices(lhs(p_EOM_c).M_2
[mu,nu]) = simplify(SumOverRepeatedIndices(rhs(p_EOM_c).M_2[mu,
nu])):
decomposed_p_EOM_c[3]:= SumOverRepeatedIndices(lhs(p_EOM_c).M_3
[mu,nu]) = simplify(SumOverRepeatedIndices(rhs(p_EOM_c).M_3[mu,
nu])):
decomposed_p_EOM_c[4]:= SumOverRepeatedIndices(lhs(p_EOM_c).M_4
[mu,nu]) = simplify(SumOverRepeatedIndices(rhs(p_EOM_c).M_4[mu,
nu])):
decomposed_p_EOM_c[5]:= SumOverRepeatedIndices(lhs(p_EOM_c).M_5
[mu,nu]) = simplify(SumOverRepeatedIndices(rhs(p_EOM_c).M_5[mu,
nu])):
decomposed_p_EOM_c[6]:= SumOverRepeatedIndices(lhs(p_EOM_c).M_6
[mu,nu]) = simplify(SumOverRepeatedIndices(rhs(p_EOM_c).M_6[mu,
nu])):
```

```
> #This is the spatial Fourier transform of the linearized diffeomorphism constraint; equation 5.23. This will be decomposed also.
Define(C_c[~nu]= SumOverRepeatedIndices(k*e_3[mu].p[~mu,~nu]) + k*p/(12*a^3)*SumOverRepeatedIndices((e[~mu,~beta].e[~nu,~alpha] + e[~mu,~alpha].e[~nu,~beta]).(e_3[alpha].h[beta,mu] + e_3[mu].h[alpha,beta] - e_3[beta].h[mu,alpha]))):
C_c[~nu,Matrix]:
```

Defined objects with tensor properties **(3.2)**

```
> longitudinal_ldc_c:=expand(simplify(C_c[~3]=0)); #Longitudinal part of the linearized diffeomorphism constraint.This is the same as equation 5.48.
transverse_ldc_c:=expand(simplify(C_c[~1]=0)); #One transverse part of the linearized diffeomorphism constraint. For the second transverse part, replace the subscript 6 by 5. This is the same as equation 5.49
```

$$longitudinal\_ldc\_c := \frac{\sqrt{3}\, k \sqrt{2}\, p2}{3} + \frac{\sqrt{3}\, k\, p1}{3} + \frac{\sqrt{3}\, k \sqrt{2}\, h2\, p}{18\, a^3} + \frac{\sqrt{3}\, k\, h1\, p}{18\, a^3} = 0$$

$$transverse\_ldc\_c := \frac{\sqrt{2}\, p6\, k}{2} + \frac{\sqrt{2}\, k\, p\, h6}{12\, a^3} = 0 \qquad \textbf{(3.3)}$$

```
> tt_p_c:=expand(decomposed_p_EOM_c[3]); #One set of tensor mode equations. For the second set, replace the label 3 by 4 i.e the two set of equations are identical; equations 5.42-5.43.
tt_h_c:=expand(decomposed_metric_EOM_c[3]);
```

$$tt\_p\_c := dot\_p3 = -\frac{p\, p3}{6\, a^2} - \frac{h3\, k^2}{2\, a^3} - \frac{5\, h3\, p^2}{144\, a^5}$$

$$tt\_h\_c := dot\_h3 = 2\, a\, p3 + \frac{p\, h3}{6\, a^2} \qquad \textbf{(3.4)}$$

```
> Sc_p_c[1]:=expand(decomposed_p_EOM_c[1]); #These are the scalar mode equations; equations 5.38-5.41.
Sc_p_c[2]:=expand(decomposed_p_EOM_c[2]);
Sc_h_c[1]:=expand(decomposed_metric_EOM_c[1]);
Sc_h_c[2]:=expand(decomposed_metric_EOM_c[2]);
```

$$Sc\_p\_c_1 := dot\_p1 = -\frac{k^2 \sqrt{2}\, h2}{6\, a^3} + \frac{p\, p1}{12\, a^2} + \frac{k^2\, h1}{3\, a^3} + \frac{p^2\, h1}{144\, a^5} + \frac{\mathrm{I} \sqrt{3}\, xi\_p\, k\, p}{18\, a}$$

$$Sc\_p\_c_2 := dot\_p2 = -\frac{p\, p2}{6\, a^2} + \frac{k^2\, h2}{6\, a^3} - \frac{5\, p^2\, h2}{144\, a^5} - \frac{\sqrt{2}\, k^2\, h1}{6\, a^3} - \frac{\mathrm{I} \sqrt{3} \sqrt{2}\, xi\_p\, k\, p}{9\, a}$$

$$Sc\_h\_c_1 := dot\_h1 = \frac{2\, \mathrm{I}\, a^2 \sqrt{3}\, k\, xi\_p}{3} - a\, p1 - \frac{p\, h1}{3\, a^2}$$

$$Sc\_h\_c_2 := dot\_h2 = 2\, a\, p2 + \frac{h2\, p}{6\, a^2} + \frac{2\, \mathrm{I} \sqrt{3} \sqrt{2}\, a^2\, k\, xi\_p}{3} \qquad \textbf{(3.5)}$$

```
> Ve_p_c[1]:=expand(decomposed_p_EOM_c[6]); #One set of vector mode equations. To get the second set, replace the subscirpt 6
```

```
by 5, and the subscript 1 by 2. These are equations 5.44-5.47
Ve_h_c[1]:=expand(decomposed_metric_EOM_c[6]);
```

$$Ve\_p\_c_1 := dot\_p6 = -\frac{I\,p\,\sqrt{2}\,k\,xi\_1}{6\,a} - \frac{p\,p6}{6\,a^2} - \frac{5\,p^2\,h6}{144\,a^5}$$

$$Ve\_h\_c_1 := dot\_h6 = I\,a^2\,\sqrt{2}\,k\,xi\_1 + 2\,a\,p6 + \frac{p\,h6}{6\,a^2} \quad \textbf{(3.6)}$$

Vita

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Publications: None.
Conference Presentations: None.